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PRACTICAL TREATISE

ON THE

DIFFERENTIAL AND INTEGRAL

CALCULUS

WITH SOME OF ITS

APPLICATIONS TO MECHANICS AND

ASTRONOMY.

BY

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P R E F A C E.

THE following pages are designed to embody the course of "Calculus and its Applications," as now taught in Columbia College. This course being purely optional, is selected by those only who have a taste for mathematical studies, and it is pursued by them more with reference to its utility than as a means of mental discipline. These circumstances have given to the present work a *practical* character that can hardly fail to commend it to those who study the Calculus for the advantages it gives them in the solution of scientific problems. To meet the wants of this class of students, great care has been taken to avoid superfluous matter; the definitions have been revised and abbreviated; the demonstrations have been simplified and condensed; the rules and principles have been illustrated and enforced by numerous examples, so chosen as to familiarize the student with the use of radical and transcendental quantities; and finally, the manner of applying the Calculus has been exemplified by the solution of a variety of problems in Mechanics and Astronomy.

The method employed in developing the principles of the science is essentially that of Leibnitz. This method,

generally known as the *method of infinitesimals*, has been adopted for several reasons: *first*, it is the method adopted in all practical investigations; *second*, it is the method most easily explained and most readily comprehended; and *third*, it is a method, as will be shown in the final note, identical in results with the more commonly adopted *method of limits*, differing from it chiefly in its phraseology and in the simplicity of its results.

The author cannot conclude this prefatory note without acknowledging his obligations to those students who, from year to year, have been willing to turn aside from the attractive pursuit of classical learning to engage in the sterner study of those processes that have contributed so much to the progress of modern science. Without their interest and co-operation this book would never have been written. He would also take this opportunity to express his thanks to his distinguished colleague, Professor J. H. VAN AMRINGE, not only for many valuable suggestions made during the progress of the work, but also for much effective labor in reading and correcting the proofs as they came from the press.

COLUMBIA COLLEGE

November 24th, 1870.

Peck

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PART I.

DIFFERENTIAL CALCULUS.

I. DEFINITIONS AND INTRODUCTORY REMARKS.

Classification of Quantities.

1. THE quantities considered in Calculus are of two kinds: *constants*, which retain a fixed value throughout the same discussion, and *variables*, which admit of all possible values that will satisfy the equations into which they enter. The former are usually denoted by leading letters of the alphabet, as a, b, c , etc.; and the latter by final letters, as x, y, z , etc.; particular values of variable quantities are denoted by writing them with one or more dashes, as x', y'', z''' , etc.

Functions of one or more Variables.

2. Relations between variables are expressed by equations. In an equation between two variables, values may be assigned to one at pleasure; the resulting equation determines the corresponding values of the other. The one to which arbitrary values are assigned is called the *independent variable*, and the remaining one is said to be a *function* of the former. If an equation contain more than two variables, all but one are *independent*, and that one is a *function* of all the others. The fact that a quantity

depends on one or more variables may be expressed as follows:

$$y = f(x); z = \varphi(x, y); F(x, y, z) = 0.$$

The *first* shows that y is a function of x , the *second* that z is a function of x and y , and the *third* that x , y , and z , depend on each other, without pointing out which is a function of the other two.

Geometrical representation of a Function.

3. Every function of one variable may be represented by the ordinate of a curve, of which the variable is the corresponding abscissa. For, let y be a function of x , and suppose x to increase by insensible gradations from $-\infty$ to $+\infty$. For each value of x there will be one or more values of y , and these, if real, will determine the position of a point with respect to two rectangular axes. These points make up a curve, at every point of which the relation between the ordinate and abscissa is the same as that between the function and independent variable. This curve is called *the curve of the function*.

For values of x that give imaginary values of y there are no points; for those that give more than one real value of y there is a corresponding number of points.

In a similar manner it may be shown that a function of two variables represents the ordinate of a surface of which the variables are corresponding abscissas.

Differentials and Differentiation.

4. Of two quantities, that is the *less* whose value is nearer to $-\infty$, and that is the *greater* whose value is nearer to $+\infty$. A quantity is said to *increase* when it

approaches $+\infty$, and to *decrease* when it approaches $-\infty$. The ordinates of a curve originate from the axis of X and of any two, that is the greater whose extremity is nearer $+\infty$, and that is the less whose extremity is nearer $-\infty$; in like manner of two abscissas, that is the greater whose extremity is nearer $+\infty$, and that the less whose extremity is nearer $-\infty$.

In what follows we shall suppose the independent variable to increase by the continued addition of a *constant* but *infinitely small* increment. For every change in the value of the variable there is a corresponding change in the value of the function. In some cases, as the variable increases, the function increases; it is then said to be an *increasing function*: in other cases the function decreases as the variable increases; it is then said to be a *decreasing function*. In all cases, the *change in value* is called an *increment*; for increasing functions the increment is *positive*, and for decreasing functions it is *negative*. The increment of the function is always infinitely small, but it is not constant, except in particular cases.

The infinitely small increment of the independent variable is called the *differential of the variable*, and the corresponding increment of the function is called the *differential of the function*. Hence, *the differential of a quantity is the difference between two consecutive values of that quantity*. It is to be observed that the difference is always found by taking the first value from the second.

The operation of finding a differential is called *differentiation*. The object of the *differential calculus* is to explain the methods of differentiating functions.

Geometrical Illustration.

5. Let KL be a curve in the plane of the rectangular axes OX and OY , and let OA and OB be two abscissas differing from each other by an infinitely small quantity AB . Through A and B draw ordinates to the curve, and let PR be parallel to OX . OA and OB are consecutive abscissas, AP and BQ are consecutive ordinates, and P and Q are consecutive points of the curve.

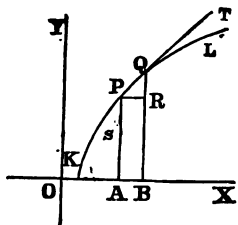


Fig. 1.

The part of the curve PQ does not differ sensibly from a straight line, and if it be prolonged toward T , the line PT is tangent to the curve at P . If we denote any abscissa OA by x , and the corresponding ordinate by y , we have,

$$y = f(x).$$

The line AB is the differential of the independent variable, denoted by the symbol dx ; RQ is the differential of the function, denoted by dy ; and PQ is the differential of the curve KL , denoted by ds .

The right angled triangle, RPQ , gives the relation $ds = \sqrt{dx^2 + dy^2}$. Denoting the angle RPQ by θ , we have, from trigonometry,

$$\tan \theta = \frac{dy}{dx}; \quad \sin \theta = \frac{dy}{ds} = \frac{dy}{\sqrt{dx^2 + dy^2}};$$

$$\text{and, } \cos \theta = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2 + dy^2}}.$$

Infinites and Infinitesimals.

6. A quantity is *infinitely great* with respect to another, when the quotient of the former by the latter is greater than any assignable number, and *infinitely small* with respect to it, when the quotient is less than any assignable number. If the term of comparison is *finite*, quantities of the former class are called *infinites*, and those of the latter *infinitesimals*.

Infinites and infinitesimals are of different orders. Let us assume the series,

$$\dots \frac{a}{x^3}, \frac{a}{x^2}, \frac{a}{x}, a, ax, ax^2, ax^3 \dots$$

In which a is a finite constant and x variable. If we suppose x to increase, the terms preceding a will diminish, and those following it will increase; when x becomes greater than any assignable quantity, $\frac{a}{x}$ becomes infinitely small with respect to a , and because each term bears the same relation to the one that follows it, every term in the series is infinitely small with respect to the following one, and infinitely great with respect to the preceding one. The quantity ax being *infinitely great* with respect to a finite quantity, is called an infinite of the *first* order; ax^2 , ax^3 , etc., are infinites of the *second*, *third*, etc., orders. The quantity $\frac{a}{x}$ being infinitely small with respect to a finite quantity is called an infinitesimal of the *first* order; $\frac{a}{x^2}$, $\frac{a}{x^3}$, etc., are infinitesimals of the *second*, *third*, etc., orders.

It is to be observed that the product of two infinitesimals of the *first* order, is an infinitesimal of the *second*

order. For, let x and y be infinitely small with respect to 1, we shall have,

$$1 : x :: y : xy.$$

Hence, xy bears the same relation to y that x does to 1, that is, it is infinitely small with respect to an infinitesimal of the first order; it is therefore an infinitesimal of the second order. The product of three infinitesimals of the first order is an infinitesimal of the third order, and so on. In general, the product of an infinitesimal of the m^{th} order by one of the n^{th} order, is an infinitesimal of the $(m + n)^{\text{th}}$ order. The product of a finite quantity by an infinitesimal of the n^{th} order, is an infinitesimal of the n^{th} order.

From the nature of an infinite quantity, its value will not be sensibly changed by the addition or subtraction of a finite quantity. A finite quantity may therefore be disregarded in comparison with an infinite quantity. For a like reason an infinitesimal may be disregarded in comparison with a finite quantity, or with an infinitesimal of a lower order. Hence, whenever an infinitesimal is connected, by the sign of addition, or subtraction, with a finite quantity, or with an infinitesimal of a lower order, it may be suppressed without affecting the value of the expression into which it enters.

General method of Differentiation.

7. In order to find the differential of a function, we give to the independent variable its infinitely small increment, and find the corresponding value of the function; from this we subtract the preceding value and reduce the result to its simplest form; we then suppress all infinitesimals which are added to, or subtracted from, those of a lower order, and the result is the differential required.

This method of proceeding is too long for general use, and is only employed in deducing rules for differentiation.

II. DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

Definition of an Algebraic Function.

* 8. An *algebraic* function is one in which the relation between the function and its variable can be expressed by the *ordinary* operations of algebra, that is, *addition, subtraction, multiplication, division, formation of powers denoted by constant exponents, and extraction of roots indicated by constant indices*. Thus,

$y^2 = 2px$, $y = ax^2 - \sqrt{bx}$, and $\sqrt{y} = \sqrt[3]{a^2x - bx^2}$,
are algebraic functions.

Differential of a Polynomial.

9. Let a and c be constants, and r, s, t , functions of x ; assume

$$y = ar + s - t + c; \dots\dots (1)$$

in which y denotes the polynomial in the second member, and is therefore a function of x .

If we give to x the increment dx , the functions y, r, s , and t , receive corresponding increments dy, dr, ds, dt , and we have,

$$y + dy = a(r + dr) + (s + ds) - (t + dt) + c \dots\dots (2)$$

subtracting (1) from (2), we have,

$$dy = adr + ds - dt \dots\dots (3)$$

Hence, to differentiate a polynomial, *differentiate each term separately and take the algebraic sum of the results*.

Comparing (1) and (3) we see, *first*, that a *constant factor* remains unchanged, and *secondly*, that a *constant term* disappears by differentiation.

Differential of a Product.

10. Let r and s be functions of x . Placing their product equal to y , we have,

$$y = rs \dots\dots (1)$$

Giving to x the increment dx , we have, as before,

$$y + dy = (r + dr)(s + ds) = rs + rds + sdr + drds. \dots (2)$$

Subtracting (1) from (2) and suppressing $drds$, which is an infinitesimal of the second order, we have, after replacing y by its value rs ,

$$d(rs) = rds + sdr \dots\dots (3)$$

Hence, to differentiate the product of two functions, *multiply each by the differential of the other, and take the algebraic sum of the results.*

If we suppose $r = tw$, we have, from the rule,

$$dr = t dw + w dt;$$

which substituted in (3), gives,

$$d(stw) = twds + stdw + swdt \dots\dots (4)$$

In like manner the principle may be extended to the product of any number of functions. Hence, to differentiate the product of any number of functions, *multiply the differential of each by the continued product of all the others, and take the algebraic sum of the results.*

COR.—If we divide both members of (4) by stw , we have,

$$\frac{d(stw)}{stw} = \frac{ds}{s} + \frac{dt}{t} + \frac{dw}{w} \dots\dots (5)$$

and similarly, where there are a greater number of factors. Hence, *the differential of a product divided by that product, is equal to the sum of the quotients obtained by dividing the differential of each factor by that factor.*

Differential of a Fraction.

11. Let s and t be functions of x , and assume,

$$y = \frac{s}{t} \dots \dots (1)$$

Giving to x the increment dx , we have,

$$y + dy = \frac{s + ds}{t + dt} \dots \dots (2)$$

Subtracting (1) from (2),

$$dy = \frac{s + ds}{t + dt} - \frac{s}{t} = \frac{tds - sdt}{t^2 + tdt}.$$

Replacing y by its value $\frac{s}{t}$, and suppressing tdt in comparison with t^2 , we have,

$$d\left(\frac{s}{t}\right) = \frac{tds - sdt}{t^2} \dots \dots (3)$$

Hence, the differential of a fraction is equal to *the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.*

If either term of the fraction be constant, its differential will be 0. When the denominator is constant, formula (3) becomes,

$$d\left(\frac{s}{c}\right) = \frac{ds}{c}; \dots \dots (4)$$

when the numerator is constant it becomes,

$$d\left(\frac{c}{t}\right) = -\frac{cdt}{t^2} \dots \dots (5)$$

Differential of a Power.

12. Let s be a function of x , and m a constant. Assume

$$y = s^m \dots \dots (1)$$

Giving to x the increment dx , we have,

$$\begin{aligned} y + dy &= (s + ds)^m = s^m + ms^{m-1}ds \\ &+ \frac{m(m-1)}{1.2} s^{m-2}ds^2 + (\text{etc.}) \dots \dots (2) \end{aligned}$$

In which all the terms of the development, after the second, contain the square, or some higher power, of ds . Subtracting (1) from (2), we have,

$$dy = ms^{m-1}ds + (\text{etc.})ds^2.$$

Suppressing all the terms of the second member, after the first, in accordance with the principle laid down in Art. 6, and replacing y by its value, we have,

$$d(s^m) = ms^{m-1}ds \dots \dots (3)$$

Hence, to differentiate any power of a function, *diminish the exponent of the function by 1, and multiply the result by the primitive exponent and the differential of the function.*

Differential of a Radical.

13. Let s be a function of x , and assume,

$$y = \sqrt[n]{s} \dots \dots (1)$$

We may write (1) in the form,

$$y = s^{\frac{1}{n}} \dots \dots (2)$$

Differentiating (2), as in the last article, we have,

$$dy = \frac{1}{n} s^{\frac{1}{n}-1} ds = \frac{1}{n} s^{\frac{1-n}{n}} ds \dots \dots (3)$$

Replacing y by its value, and remembering that

$$s^{1-n} = \frac{1}{s^{n-1}}, \text{ we have,}$$

$$d\sqrt[n]{s} = \frac{1}{n} \frac{ds}{\sqrt[n]{s^{n-1}}} \dots \dots (4)$$

That is, the differential of a radical of the n^{th} degree is equal to *the differential of the quantity under the radical sign, divided by n times the $(n-1)^{\text{th}}$ power of the radical.*

COR.—If $n = 2$, we have,

$$d\sqrt{s} = \frac{ds}{2\sqrt{s}} \dots \dots (5)$$

That is, the differential of the square root of a quantity is equal to *the differential of the quantity under the radical sign, divided by twice the radical.*

The preceding rules are sufficient to differentiate any algebraic function whatever.

EXAMPLES.

1. Let
- $y = 5x^3$
- .

This is the product of the function x^3 by the constant 5; hence, (Art. 9),

$$dy = 5d(x^3) = 5 \times 3x^2 dx = 15x^2 dx \dots \dots (\text{Art. 12}).$$

2. Let
- $y = x^3 + 2x^2 + 3x + 4$
- .

This is a polynomial. Hence, from Art. 9,

$$dy = 3x^2 dx + 4x dx + 3dx.$$

3. Let
- $y = (a + bx)^3$
- .

Considering $a + bx$ as a single quantity, we have, (Art. 12),

$$dy = 3(a + bx)^2 d(a + bx) = 3(a + bx)^2 b dx = 3b(a + bx)^2 dx.$$

4. Let
- $y = (a + bx^2)^n$
- .

$$dy = n(a + bx^2)^{n-1} d(a + bx^2) = 2bnx(a + bx^2)^{n-1} dx.$$

5. Let
- $y = \sqrt{a^2 - x^2} = (a^2 - x^2)^{\frac{1}{2}}$
- .

$$dy = \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x dx = -\frac{x dx}{\sqrt{a^2 - x^2}}.$$

6. Let
- $y = (a + x)(b + 2x^2)$
- .

By Art. 10, we have,

$$\begin{aligned} dy &= (a + x)d(b + 2x^2) + (b + 2x^2)d(a + x) \\ &= (a + x)4x dx + (b + 2x^2)dx; \text{ or, } dy \\ &= (6x^2 + 4ax + b)dx. \end{aligned}$$

7. Let $y = \frac{ax}{a^2 + x^2}$.

By Art. 11, we have,

$$dy = \frac{(a^2 + x^2)d(ax) - axd(a^2 + x^2)}{(a^2 + x^2)^2} = \frac{a(a^2 - x^2)}{(a^2 + x^2)^2} dx.$$

8. Let $y = \sqrt{x + \sqrt{a^2 + x^2}}$.

$$dy = d\left[x + (a^2 + x^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} = \frac{1}{2}\left[x + (a^2 + x^2)^{\frac{1}{2}}\right]^{-\frac{1}{2}}$$

$$\times d\left[x + (a^2 + x^2)^{\frac{1}{2}}\right]$$

$$\text{but, } d\left[x + (a^2 + x^2)^{\frac{1}{2}}\right] = dx + xdx(a^2 + x^2)^{-\frac{1}{2}}$$

$$= \left[x + (a^2 + x^2)^{\frac{1}{2}}\right](a^2 + x^2)^{-\frac{1}{2}} dx,$$

$$\therefore dy = \frac{\sqrt{x + \sqrt{a^2 + x^2}}}{2\sqrt{a^2 + x^2}} dx.$$

9. $y = ax^{\frac{5}{2}} - bx^{\frac{3}{2}} \pm c$. Ans. $dy = (\frac{5}{2}ax^{\frac{3}{2}} - \frac{3}{2}bx^{\frac{1}{2}})dx$.

10. $y = ax^{-\frac{4}{3}} - bx^{-\frac{2}{3}} \pm c$.

$$\text{Ans. } dy = -\left(\frac{4}{3}ax^{-\frac{7}{3}} - \frac{2}{3}bx^{-\frac{5}{3}}\right)dx.$$

11. $y = (a^2 - x^2)^{\frac{3}{2}}$. Ans. $dy = -4x(a^2 - x^2)dx$.

12. $y = (2ax - x^2)^3$. Ans. $dy = 6(a - x)(2ax - x^2)^2 dx$.

$$13. y = (a + bx^n)^m. \text{ Ans. } dy = mnbx^{n-1}(a + bx^n)^{m-1}dx.$$

$$14. y = (2ax + x^2)^n. \text{ Ans. } dy = 2n(a + x)(2ax + x^2)^{n-1}dx.$$

$$15. y = x(a + x)(a^2 + x^2).$$

$$\text{Ans. } dy = (a^3 + 2a^2x + 3ax^2 + 4x^3)dx.$$

$$16. y = (a + x)^m(b + x)^n.$$

$$\text{Ans. } dy = (a + x)^m(b + x)^n \left[\frac{m}{a + x} + \frac{n}{b + x} \right] dx.$$

$$17. y = \frac{a - x}{a + x}.$$

$$\text{Ans. } dy = -\frac{2a}{(a + x)^2}dx.$$

$$18. y = \frac{(x + 4)^2}{x + 3}.$$

$$\text{Ans. } dy = \frac{(x + 2)(x + 4)}{(x + 3)^2}dx.$$

$$19. y = \frac{x^2 - x + 1}{x^2 + x - 1}.$$

$$\text{Ans. } dy = \frac{2x(x - 2)}{(x^2 + x - 1)^2}dx$$

$$20. y = \sqrt{a + x}.$$

$$\text{Ans. } dy = \frac{dx}{2\sqrt{a + x}}.$$

$$21. y = \sqrt{1 + x^2}$$

$$\text{Ans. } dy = \frac{xdx}{\sqrt{1 + x^2}}.$$

$$22. y = \sqrt{ax^2 + bx + c}.$$

$$\text{Ans. } dy = \frac{(2ax + b)dx}{2\sqrt{ax^2 + bx + c}}.$$

$$23. y = (a - x)\sqrt{a + x}.$$

$$\text{Ans. } dy = -\frac{(a + 3x)dx}{2\sqrt{a + x}}.$$

$$24. y = (a + x)\sqrt{a - x}.$$

$$\text{Ans. } dy = \frac{(a - 3x)dx}{2\sqrt{a - x}}.$$

$$\star 25. y = (1 - x^{\frac{1}{2}} + x^{\frac{3}{2}})^{\frac{1}{2}}. \quad \text{Ans. } dy = \frac{1}{8} \frac{(4x^{\frac{1}{2}} - 3)dx}{x^{\frac{1}{2}}(1 - x^{\frac{1}{2}} + x^{\frac{3}{2}})^{\frac{1}{2}}}.$$

$$26. y = (a^2 - x^2)\sqrt{a+x}. \quad \text{Ans. } dy = [\frac{1}{2}(a - 5x)\sqrt{a+x}]dx.$$

$$\times 27. y = \frac{a+x}{\sqrt{a-x}}. \quad \text{Ans. } dy = \frac{(3a-x)dx}{2(a-x)^{\frac{3}{2}}}.$$

$$28. y = \frac{\sqrt{a+x}}{\sqrt{a-x}}. \quad \text{Ans. } dy = \frac{adx}{(a-x)\sqrt{a^2-x^2}}.$$

$$\star 29. y = \sqrt{x - \sqrt{a^2 - x^2}}. \quad \text{Ans. } dy = \frac{(x + \sqrt{a^2 - x^2})dx}{2\sqrt{a^2 - x^2}(x - \sqrt{a^2 - x^2})^{\frac{1}{2}}}.$$

$$\times 30. y = x(1+x^2)(1-x^2)^{\frac{1}{2}}. \quad \text{Ans. } dy = \frac{(1+x^2-4x^4)dx}{\sqrt{1-x^2}}.$$

$$31. y = (1+2x^2)(1+4x^2). \quad \text{Ans. } dy = 4x(1+3x+10x^3)dx.$$

$$< 32. y = \frac{x}{x + \sqrt{1-x^2}}. \quad \text{Ans. } dy = \frac{dx}{2x(1-x^2) + \sqrt{1-x^2}}.$$

$$33. y = (1+x)\sqrt{1-x}. \quad \text{Ans. } dy = \frac{(1-3x)dx}{2\sqrt{1-x}}.$$

$$34. y = \frac{x^2}{\sqrt{(1-x^2)^2}}. \quad \text{Ans. } dy = \frac{3x^2dx}{(1-x^2)^{\frac{3}{2}}}.$$

$$35. y = \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}} \quad \text{Ans. } dy = -\frac{dx}{2(1 + \sqrt{x})\sqrt{x - x^2}}.$$

$$36. y = \sqrt{2x - 1 - \sqrt{2x - 1 - \sqrt{2x - 1 - \text{etc.}, ad inf.}}}$$

$$\text{Squaring, } y^2 = 2x - 1 - \sqrt{2x - 1 - \sqrt{2x - 1 - \text{etc.}, ad inf.}}$$

$$\text{Hence, } y^2 = 2x - 1 - y, \text{ or } y^2 + y = 2x - 1.$$

$$\text{Solving, } y = -\frac{1}{2} \pm \sqrt{2x - 1 + \frac{1}{4}} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{8x - 3}.$$

$$\therefore dy = \pm \frac{1}{2} \frac{4dx}{\sqrt{8x - 3}} = \pm \frac{2dx}{\sqrt{8x - 3}} \quad \text{Ans.}$$

III. DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

Definition of a Transcendental Function.

14. A *transcendental function* is one in which the relation between the function and its variable cannot be expressed by the *ordinary* operations of algebra.

Transcendental functions are divided into *three* classes: *Logarithmic*, in which the relation is expressed by logarithmic symbols; *Exponential*, in which the variable enters an exponent; and *Circular*, in which the relation is expressed by means of trigonometric symbols. Thus,

$$y = \log(ax^2 + b), \text{ is a } \textit{logarithmic} \text{ function,}$$

$$y = (ax^2 + c)^{2x-c}, \text{ is an } \textit{exponential} \text{ function,}$$

$$\text{and } y = \sin^2 x - 2 \tan x, \text{ is a } \textit{trigonometric} \text{ function.}$$

Differential of a Logarithm.

15. It is shown in Algebra that

$$\log (1+y)=M\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\text{etc.}\right) \ldots \ldots (1)$$

In which M is the modulus of the system and y any quantity whatever.

Substitute for y the quantity $\frac{ds}{s}$, s being a function of x , and we have, after reduction,

$$\log \left(\frac{s+ds}{s}\right)=M\left(\frac{ds}{s}-\frac{ds^2}{2s^2}+\frac{ds^3}{3s^3}-\text{etc.}\right) \ldots \ldots (2)$$

But the logarithm of a quotient, is equal to the logarithm of the dividend, diminished by the logarithm of the divisor; changing the form of the first member and suppressing all the terms in the second member, after the first, (Art. 6), we have,

$$\log (s+ds)-\log s=M \frac{ds}{s} \ldots \ldots (3)$$

But the first member is the difference between two consecutive values of $\log s$; it is therefore the differential of the logarithm of s ; hence,

$$d(\log s)=M \frac{ds}{s} \ldots \ldots (4)$$

That is, the differential of the logarithm of a quantity is equal to *the modulus into the differential of the quantity, divided by the quantity.*

In analysis, Napierian logarithms are almost always used.

Denoting the Napierian logarithm by l , and remembering that the modulus of this system is 1, we have,

$$d(ls) = \frac{ds}{s} \dots\dots (5)$$

The base of the Napierian system is represented by the letter e , and its logarithm in that system is equal to 1.

✕

Differential of an Exponential Function.

16. Let a be a constant quantity, s any function of x , and assume

$$y = a^s \dots\dots (1)$$

Taking the logarithms of both members of (1), we have,

$$ly = sla \dots\dots (2)$$

Differentiating (2), we have,

$$\frac{dy}{y} = la \times ds \dots\dots (3)$$

Replacing y by its value, and reducing, we have,

$$d(a^s) = a^s lads \dots\dots (4)$$

Hence, to differentiate a quantity formed by raising a constant quantity to a power denoted by a variable exponent, *multiply the quantity by the logarithm of the root and the differential of the exponent.*

Again, let us have,

$$y = t^s \dots\dots (1)$$

in which both t and s are functions of x . Taking the logarithm of both members, we have,

$$ly = slt \dots\dots (2)$$

Whence, by differentiating both members,

$$\frac{dy}{y} = t ds + s \frac{dt}{t} \dots \dots (3)$$

Replacing y by its value, clearing of fractions and reducing, we have,

$$d(t^s) = t^s t ds + s t^{s-1} dt \dots \dots (4)$$

That is, to differentiate a quantity formed by raising a variable quantity to a power whose exponent is variable, *differentiate first as though the root alone were variable, then as though the exponent alone were variable, and take the sum of the results.*

EXAMPLES.

1. Let $y = l\left(\frac{a+x}{a-x}\right) = l(a+x) - l(a-x);$

$$\begin{aligned} \therefore dy &= \frac{d(a+x)}{a+x} - \frac{d(a-x)}{a-x} = \frac{dx}{a+x} + \frac{dx}{a-x} \\ &= \frac{2adx}{a^2 - x^2}. \text{ Ans.} \end{aligned}$$

2. Let $y = l\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}l(1+x) - \frac{1}{2}l(1-x);$

$$\therefore dy = \frac{1}{2} \frac{dx}{1+x} + \frac{1}{2} \frac{dx}{1-x} = \frac{dx}{1-x^2}. \text{ Ans.}$$

3. Let $y = (a)^{e^x}; \therefore ly = e^x la, \text{ and } \frac{dy}{y} = la e^x le dx.$

But, $le = 1, \therefore dy = (a)^{e^x} e^x la dx. \text{ Ans.}$

4. Let $y = l(lx) = lx$,

By the rule, $dy = \frac{d(lx)}{lx} = \frac{dx}{x}$. *Ans.*

5. $y = x l(a + x)$. *Ans.* $dy = l(a + x)dx + \frac{xdx}{a + x}$

6. $y = l(a + x)^2 = 2l(a + x)$. *Ans.* $dy = \frac{2dx}{a + x}$

7. $y = e^x(x - 1)$. *Ans.* $dy = e^x x dx$

8. $y = e^x(x^2 - 2x + 2)$. *Ans.* $dy = e^x x^2 dx$

9. $y = \frac{e^x - 1}{e^x + 1}$. *Ans.* $dy = \frac{2e^x}{(e^x + 1)^2} dx$

10. $y = e^x lx$. *Ans.* $dy = e^x \left(lx + \frac{1}{x} \right) dx$

11. $y = l(x + a + \sqrt{2ax + x^2})$. *Ans.* $dy = \frac{dx}{\sqrt{2ax + x^2}}$

12. $y = \frac{e^x}{1 + x}$. *Ans.* $dy = \frac{xe^x dx}{(1 + x)^2}$

13. $y = l \frac{x}{\sqrt{x^2 + 1} + x}$. *Ans.* $dy = \frac{dx}{x} - \frac{dx}{\sqrt{x^2 + 1}}$

14. $y = e^x \sqrt{\frac{1+x}{1-x}}$. *Ans.* $dy = e^x \frac{2-x^2}{1-x^2} \sqrt{\frac{1+x}{1-x}} dx$

15. $y = x^x$. *Ans.* $dy = x^x (lx + 1) dx$

16. $y = a^{lx}$. *Ans.* $dy = a^{lx} la x^{-1} dx$

Differentials of Circular Functions.

17. Assume the equation,

$$y = \sin x \dots\dots (1)$$

Giving to x the increment dx , and developing, we have,

$$\begin{aligned} y + dy &= \sin(x + dx) \\ &= \sin x \cos dx + \cos x \sin dx \dots\dots (2) \end{aligned}$$

Because the arc dx is infinitely small, its *cosine* is equal to 1, and its *sine* is equal to the arc itself. Hence,

$$y + dy = \sin x + \cos x dx \dots\dots (3)$$

Subtracting (1) from (3), and replacing y by its value, we have,

$$d(\sin x) = \cos x dx \dots\dots (a)$$

Equation (a) is true for all values of x ; we may therefore replace x by $(90^\circ - x)$; this gives,

$$d[\sin(90^\circ - x)] = \cos(90^\circ - x)d(90^\circ - x) \dots\dots (4)$$

Reducing, we have,

$$d(\cos x) = -\sin x dx \dots\dots (b)$$

We have, from trigonometry, the relation,

$$\tan x = \frac{\sin x}{\cos x} \dots\dots (5)$$

Differentiating both members, we have,

$$d(\tan x) = \frac{\cos x d(\sin x) - \sin x d(\cos x)}{\cos^2 x}.$$

Performing indicated operations, and reducing by the relation, $\sin^2 x + \cos^2 x = 1$, we have,

$$d(\tan x) = \frac{dx}{\cos^2 x} \dots\dots (c)$$

Replacing x , in formula (c), by $(90^\circ - x)$ and reducing, we have,

$$d(\cot x) = -\frac{dx}{\sin^2 x} \dots\dots (d)$$

We have, from trigonometry,

$$\text{ver-sin } x = 1 - \cos x \dots\dots (6)$$

$$\text{co-versin } x = 1 - \sin x \dots\dots (7)$$

Differentiating (6) and (7), we have,

$$d(\text{ver-sin } x) = \sin x dx \dots\dots (e)$$

$$d(\text{co-versin } x) = -\cos x dx \dots\dots (f)$$

We have, from trigonometry,

$$\sec x = \frac{1}{\cos x} \dots\dots (8)$$

$$\text{cosec } x = \frac{1}{\sin x} \dots\dots (9)$$

Differentiating (8) and (9), we have, after reduction,

$$d(\sec x) = \tan x \sec x dx \dots\dots (g)$$

$$d(\text{cosec } x) = -\cot x \text{ cosec } x dx \dots\dots (h)$$

The *lettered* formulas, from (a) to (h) inclusive, are sufficient for the differentiation of any direct circular function.

EXAMPLES.

1. Let $y = \sin mx$.

$$dy = \cos mx d(mx) = m \cos mx \cdot dx.$$

2. Let $y = \sin^m x$.

$$dy = m \sin^{m-1} x d\sin x = m \sin^{m-1} x \cos x dx.$$

3. Let $y = \sin 2x \cos x$.

$$\begin{aligned} dy &= (d\sin 2x)(\cos x) + (d\cos x)(\sin 2x) \\ &= 2\cos 2x \cos x dx - \sin 2x \sin x dx. \end{aligned}$$

4. Let $y = l(\sin^2 x) = 2l \sin x$.

$$dy = 2 \frac{d\sin x}{\sin x} = \frac{2\cos x}{\sin x} dx = 2\cot x dx.$$

5. Let $y = l\left(\frac{1 + \cos x}{1 - \cos x}\right)^{\frac{1}{2}} = \frac{1}{2}l(1 + \cos x) - \frac{1}{2}l(1 - \cos x)$.

$$dy = -\frac{1}{2} \frac{\sin x dx}{1 + \cos x} - \frac{1}{2} \frac{\sin x dx}{1 - \cos x} = -\frac{\sin x dx}{1 - \cos^2 x} = -\frac{dx}{\sin x}.$$

6. Let $y = e^x \cos x$.

$$dy = e^x d(\cos x) + \cos x d(e^x) = e^x (\cos x - \sin x) dx.$$

7. $y = x \sin x + \cos x$.

Ans. $dy = x \cos x dx$.

8. $y = 2x \sin x + (2 - x^2) \cos x$.

Ans. $dy = x^2 \sin x dx$.

9. $y = \tan x - x$.

Ans. $dy = \tan^2 x dx$.

10. $y = e^{\cos x} \sin x$.

Ans. $dy = e^{\cos x} (\cos x - \sin^2 x) dx$.

11. $y = 2l(\sin x) + \operatorname{cosec} x$. Ans. $dy = \left(2\cot x - \frac{\cot x}{\sin x}\right) dx$.

$$12. y = l(\cos x + \sqrt{-1} \sin x). \quad \text{Ans. } dy = \sqrt{-1} dx.$$

$$13. y = l\left(\frac{1 + \sin x}{1 - \sin x}\right)^{\frac{1}{2}} \quad \text{Ans. } dy = \frac{dx}{\cos x}.$$

$$14. y = l[\tan(45^\circ + \frac{1}{2}x)]. \quad \text{Ans. } dy = \frac{dx}{\cos x}.$$

$$15. y = \sin(lx). \quad \text{Ans. } dy = \frac{1}{x} \cos(lx) dx.$$

Differentials of Inverse Circular Functions.

18. It is often convenient to regard an *arc* as a function of one of its *trigonometrical lines*. Such functions are called *inverse circular functions*, and are expressed by such symbols as the following:

$$\sin^{-1}y, \cos^{-1}y, \tan^{-1}y, \text{ etc.,}$$

which are read *the arc whose sine is y, the arc whose cosine is y, the arc whose tangent is y, etc.*

Formulas for the differentiation of inverse circular functions may be deduced from the lettered formulas of the last article. Thus, from formula (a), we find,

$$dx = \frac{d(\sin x)}{\cos x} \dots \dots (1)$$

If we make $\sin x = y$, we have,

$$x = \sin^{-1}y, \text{ and } \cos x = \sqrt{1 - y^2}.$$

Substituting these in (1), we have,

$$d(\sin^{-1}y) = \frac{dy}{\sqrt{1 - y^2}} \dots \dots (a^1)$$

In like manner, from formula (b), we find,

$$dx = -\frac{d(\cos x)}{\sin x} \dots\dots (2)$$

Making $\cos x = y$, we have,

$$x = \cos^{-1}y, \text{ and } \sin x = \sqrt{1-y^2};$$

Substituting these in (2), we have,

$$d(\cos^{-1}y) = -\frac{dy}{\sqrt{1-y^2}} \dots\dots (b^1)$$

By a similar course of reasoning we find from the remaining lettered formulas of Art. 17, the corresponding formulas for inverse circular functions, as given below:

$$d(\tan^{-1}y) = \frac{dy}{1+y^2} \dots\dots (c^1)$$

$$d(\cot^{-1}y) = -\frac{dy}{1+y^2} \dots\dots (d^1)$$

$$d(\operatorname{versin}^{-1}y) = \frac{dy}{\sqrt{2y-y^2}} \dots\dots (e^1)$$

$$d(\operatorname{coversin}^{-1}y) = -\frac{dy}{\sqrt{2y-y^2}} \dots\dots (f^1)$$

$$d(\sec^{-1}y) = \frac{dy}{y\sqrt{y^2-1}} \dots\dots (g^1)$$

$$d(\operatorname{cosec}^{-1}y) = -\frac{dy}{y\sqrt{y^2-1}} \dots\dots (h^1)$$

EXAMPLES.

1. Let $y = \cos^{-1} x \sqrt{1-x^2} = \cos^{-1} \sqrt{x^2-x^4}$.

$$d \sqrt{x^2-x^4} = \frac{(1-2x^2)dx}{\sqrt{1-x^2}}$$

$$\text{and } \sqrt{1-(x^2-x^4)} = \sqrt{1-x^2+x^4},$$

$$\therefore dy = - \frac{(1-2x^2) dx}{\sqrt{(1-x^2)(1-x^2+x^4)}}$$

2. Let $y = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$.

$$\begin{aligned} dy &= d \left(\frac{x}{\sqrt{1+x^2}} \right) \div \sqrt{1 - \frac{x^2}{1+x^2}} \\ &= \frac{dx}{(1+x^2)^{\frac{3}{2}}} \div \frac{1}{(1+x^2)^{\frac{1}{2}}} = \frac{dx}{(1+x^2)} \end{aligned}$$

3. Let $y = \sin^{-1} 2x\sqrt{1-x^2}$.

$$\begin{aligned} dy &= d(2x\sqrt{1-x^2}) \div \sqrt{1-4x^2(1-x^2)} \\ &= \frac{2(1-2x^2)dx}{\sqrt{1-x^2}} \div (1-2x^2) = \frac{2dx}{\sqrt{1-x^2}}. \end{aligned}$$

4. $y = \tan^{-1} \frac{2x}{1-x^2}$.

$$\text{Ans. } dy = \frac{2dx}{1+x^2}.$$

5. $y = \sin^{-1} \frac{1-x^2}{1+x^2}$.

$$\text{Ans. } dy = \frac{2dx}{1+x^2}.$$

$$* 6. y = \sqrt{a^2 - x^2} + a \sin^{-1} \frac{x}{a}. \quad \text{Ans. } dy = \left(\frac{a - x}{a + x} \right)^{\frac{1}{2}} dx.$$

$$+ 7. y = \frac{1}{2} \operatorname{versin}^{-1} \frac{8x}{9}. \quad \text{Ans. } dy = \frac{dx}{\sqrt{9x - 4x^2}}.$$

$$8. y = \sec^{-1} 2x. \quad \text{Ans. } dy = \frac{dx}{x\sqrt{4x^2 - 1}}.$$

IV. SUCCESSIVE DIFFERENTIATION AND DEVELOPMENT OF FUNCTIONS.

Successive Differentials.

19. The differential obtained immediately from the function is *the first differential of the function*; the differential of the first differential is *the second differential of the function*; the differential of the second differential is *the third differential of the function*, and so on; differentials thus obtained are called *successive differentials*, and the operation of obtaining them is called *successive differentiation*. If a function be denoted by y , its successive differentials will be denoted by the symbols dy , d^2y , d^3y , etc. Thus, if $y = ax^3$, we have, by successive differentiation, dx being constant, $dy = 3ax^2dx$, $d^2y = 6axdx^2$, $d^3y = 6adx^3$, $d^4y = 0$.

Successive Differential Coefficients.

20. If the differential of a function be divided by the differential of the variable, the quotient is *the first differential coefficient of the function*; the differential coefficient of the first differential coefficient is the *second differential coefficient* of the function, and so on. Differential coeffi-

cients, derived in this manner, are called *successive differential coefficients*. If a function of x be denoted by y , its successive differential coefficients are denoted by the symbols, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc. Thus, if we have, as before, $y = ax^3$, we have, from the principles just explained,

$$\frac{dy}{dx} = 3ax^2; \quad \frac{d^2y}{dx^2} = 6ax; \quad \frac{d^3y}{dx^3} = 6a; \quad \frac{d^4y}{dx^4} = 0.$$

It is to be observed that the successive differential coefficients are entirely independent of the differential of the variable; so long therefore as this is infinitesimal, the differential coefficients will in no way be affected by any change in its absolute value.

EXAMPLES.

Find the successive differential coefficients of the following functions:

1. $y = ax^n$.

Ans. $\frac{dy}{dx} = nax^{n-1}; \quad \frac{d^2y}{dx^2} = n(n-1)ax^{n-2};$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)ax^{n-3}; \text{ etc.}$$

If n is a positive whole number, there will be a finite number of successive differential coefficients; otherwise their number is infinite.

2. $y = ax^3 + bx^2$.

Ans. $\frac{dy}{dx} = 3ax^2 + 2bx; \quad \frac{d^2y}{dx^2} = 6ax + 2b;$

$$\frac{d^3y}{dx^3} = 6a; \quad \frac{d^4y}{dx^4} = 0.$$

3. $y = a^x.$

Ans. $\frac{dy}{dx} = a^x \log a; \frac{d^2y}{dx^2} = a^x (\log a)^2; \frac{d^3y}{dx^3} = a^x (\log a)^3$

$$\dots \frac{d^ny}{dx^n} = a^x (\log a)^n.$$

4. $y = \sin x.$

Ans. $\frac{dy}{dx} = \cos x; \frac{d^2y}{dx^2} = -\sin x;$

$$\frac{d^3y}{dx^3} = -\cos x; \frac{d^4y}{dx^4} = \sin x; \text{ etc.}$$

5. $y = l(x + 1).$

Ans. $\frac{dy}{dx} = \frac{1}{x+1} = (x+1)^{-1}; \frac{d^2y}{dx^2} = -(x+1)^{-2};$

$$\frac{d^3y}{dx^3} = 2(x+1)^{-3}; \frac{d^4y}{dx^4} = -6(x+1)^{-4}; \text{ etc.}$$

6. $y = xe^x.$

Ans. $\frac{dy}{dx} = (x+1)e^x; \frac{d^2y}{dx^2} = (x+2)e^x$

$$\dots \frac{d^ny}{dx^n} = (x+n)e^x.$$



McLaurin's Formula.

21. McLaurin's formula, is a formula for developing a function of one variable into a series arranged according to the ascending powers of that variable, the coefficients being constant. Let y be any function of x , and assume the development,

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.} \dots (1)$$

It is required to find such values for A, B, C , etc., as will make the assumed development true for all values of x . Differentiating, and finding the successive differential coefficients of y , we have,

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \text{etc.} \dots (2)$$

$$\frac{d^2y}{dx^2} = 1.2C + 1.2.3Dx + 3.4Ex^2 + \text{etc.} \dots (3)$$

$$\frac{d^3y}{dx^3} = 1.2.3D + 1.2.3.4Ex + \text{etc.} \dots (4)$$

etc., etc., etc.

But, by hypothesis, the value of y , and consequently the values of its successive differential coefficients, are to be true for all values of x ; hence, they must be so for $x = 0$. Making $x = 0$, in equations (1), (2), (3), etc., and denoting what y becomes under that hypothesis by (y) ; what $\frac{dy}{dx}$ becomes by $\left(\frac{dy}{dx}\right)$; what $\frac{d^2y}{dx^2}$ becomes by $\left(\frac{d^2y}{dx^2}\right)$, and so on; we have,

$$(y) = A; \quad \therefore A = (y);$$

$$\left(\frac{dy}{dx}\right) = B; \quad \therefore B = \left(\frac{dy}{dx}\right);$$

$$\left(\frac{d^2y}{dx^2}\right) = 1.2C; \quad \therefore C = \frac{1}{1.2} \left(\frac{d^2y}{dx^2}\right);$$

$$\left(\frac{d^3y}{dx^3}\right) = 1.2.3.D; \quad \therefore D = \frac{1}{1.2.3} \left(\frac{d^3y}{dx^3}\right);$$

etc., etc., etc.

Substituting these values in (1), we have,

$$y = (y) + \left(\frac{dy}{dx}\right) \cdot \frac{x}{1} + \left(\frac{d^2y}{dx^2}\right) \frac{x^2}{1.2} + \left(\frac{d^3y}{dx^3}\right) \cdot \frac{x^3}{1.2.3} + \text{etc.} \quad (5)$$

which is the formula required. Hence, to develop a function of one variable in terms of that variable, find its successive differential coefficients; then make the variable equal to 0 in the function and its successive differential coefficients, and substitute the results for (y) , $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, etc., in formula (5).

Thus, let it be required to develop $\sin x$ into a series arranged with reference to x . We have,

$$y = \sin x, \quad \therefore (y) = 0$$

$$\frac{dy}{dx} = \cos x, \quad \therefore \left(\frac{dy}{dx}\right) = 1$$

$$\frac{d^2y}{dx^2} = -\sin x \quad \therefore \left(\frac{d^2y}{dx^2}\right) = 0$$

$$\frac{d^3y}{dx^3} = -\cos x, \quad \therefore \left(\frac{d^3y}{dx^3}\right) = -1$$

$$\text{etc.,} \quad \text{etc.,} \quad \text{etc.}$$

Hence,

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \text{etc.} \dots (6)$$

In like manner we find,

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{etc.} \dots (7)$$

It is to be observed that (7) may be found from (6) by differentiating and then dividing by dx .

McLaurin's Formula enables us to develop any function of one variable when it can be developed in accordance with the assumed law. But there are functions which cannot be developed according to the ascending powers of the variable; in this case the function, or some of its successive differential coefficients, become ∞ , when the variable is made equal to 0.

As a general rule, when the application of a formula gives an infinite result, the formula is inapplicable in that particular case.

EXAMPLES.

1. $y = (a + x)^n$.

$$\begin{aligned} \text{Ans. } y = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 \\ + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \text{etc.} \end{aligned}$$

2. $y = l(1 + x)$.

$$\text{Ans. } y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$$

3. $y = a^x$.

$$\text{Ans. } y = 1 + \frac{(la)}{1}x + \frac{(la)^2}{1.2}x^2 + \frac{(la)^3}{1.2.3}x^3 + \frac{(la)^4}{1.2.3.4}x^4 + \text{etc.}$$

4. $y = \sqrt{1 + x^2} = (1 + x^2)^{\frac{1}{2}}$.

Make $x^2 = z$, and develop; then replace z by its value.

$$\text{Ans. } y = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \text{etc.}$$

5. $y = e^{\sin x}$.

$$\text{Ans. } y = 1 + x + \frac{x^2}{2} - \frac{3x^4}{1.2.3.4} - \frac{8x^5}{1.2.3.4.5} - \frac{3x^6}{1.2.3.4.5.6} + \text{etc.}$$

6. $y = xe^x + e^x - 1$.

$$\text{Ans. } y = 2x + \frac{3x^2}{1.2} + \frac{4x^3}{1.2.3} + \frac{5x^4}{1.2.3.4} + \text{etc.}$$

*

Taylor's Formula.

22. Taylor's Formula is a formula for developing a function of the *sum* of two variables into a series arranged according to the ascending powers of one, with coefficients that are functions of the other.

LEMMA.

If $u' = f(x + y)$, the differential coefficient of u' will be the same, whether we suppose x to remain constant and y to vary, or y to remain constant and x to vary, for the form of the function is the same, whichever we suppose to vary; and it has been shown, in Art. 20, that the value of the differential coefficient is independent of the value of the differential of the variable. Hence, if $x + y$ be increased either by dx or by dy , and the differential coefficient taken, the result will be the same, which was to be shown.

Let u' be a function of $(x + y)$, and assume the development,

$$u' = P + Qx + Rx^2 + Sx^3 + Tx^4 + \text{etc.} \dots \dots (1)$$

in which P, Q, R , etc., are functions of y . It is required to find such values of P, Q, R , etc., as will make equation

(1) true for all values of both x and y . Since the assumed development is to be true for all values of x and y , it must be so for $x = 0$. Making $x = 0$ in (1), and denoting what u' becomes under this hypothesis, by u , we find,

$$P = u.$$

That is, P is what the original function becomes, when the *leading* variable is made equal to 0.

Finding the differential coefficient of u' , under the supposition that x is constant and y variable, we have,

$$\frac{du'}{dy} = \frac{dP}{dy} + \frac{dQ}{dy}x + \frac{dR}{dy}x^2 + \frac{dS}{dy}x^3 + \text{etc.} \dots \dots (2)$$

Again, finding the differential coefficient of u' , on the supposition that y is constant and x variable, we have,

$$\frac{du'}{dx} = Q + 2Rx + 3Sx^2 + 4Tx^3 + \text{etc.} \dots \dots (3)$$

But, the first members of (2) and (3) are equal, by the lemma; hence their second members are also equal. If we place them equal we have an identical equation, because it is true for all values of x ; and consequently the coefficients of the like powers of x in the two members are equal to each other. Placing their coefficients equal, we have the following results:

$$\begin{array}{lll} Q = \frac{dP}{dy} & \therefore Q = \frac{du}{dy}, \\ 2R = \frac{dQ}{dy} & \therefore R = \frac{1}{1.2} \frac{d^2u}{dy^2}, \\ 3S = \frac{dR}{dy} & \therefore S = \frac{1}{1.2.3} \frac{d^3u}{dy^3}, \\ \text{etc.,} & \text{etc.,} & \text{etc.} \end{array}$$

Substituting the values of P , Q , R , S , etc., in (1), we have,

$$u' = u + \frac{du}{dy} \cdot \frac{x}{1} + \frac{d^2u}{dy^2} \cdot \frac{x^2}{1.2} + \frac{d^3u}{dy^3} \cdot \frac{x^3}{1.2.3} \\ + \frac{d^4u}{dy^4} \cdot \frac{x^4}{1.2.3.4} + \text{etc.} \dots \dots (4)$$

which is the required formula. Hence, to develop any function of the sum of two variables, we make the leading variable equal to 0, and find the successive differential coefficients of the result; then substitute them in the formula.

Thus, let it be required to develop $(x + y)^n$ into a series arranged according to the ascending powers of y . Making $y = 0$, we have,

$$u = x^n,$$

$$\frac{du}{dx} = nx^{n-1},$$

$$\frac{d^2u}{dx^2} = n(n-1)x^{n-2},$$

$$\frac{d^3u}{dx^3} = n(n-1)(n-2)x^{n-3},$$

$$\text{etc.,} \quad \text{etc.,} \quad \text{etc.}$$

Substituting these in (4), we have,

$$(x + y)^n = x^n + \frac{n}{1}x^{n-1}y + \frac{n.n-1}{1.2}x^{n-2}y^2 \\ + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + \text{etc.}$$

This is the binomial formula, in which n is any constant.

The formula is applicable to every function of the sum of two variables; but it sometimes happens that certain values of the variable entering the coefficients make the first term or some of its successive differential coefficients *infinite*; for these particular values, the function cannot be expressed by a series of the proposed form.

EXAMPLES.

Develop the following functions in terms of y .



1. $u' = \sin(x + y)$.

Making $y = 0$, we have, $u = \sin x$;

$$\therefore \frac{du}{dx} = \cos x; \quad \frac{d^2u}{dx^2} = -\sin x; \text{ etc.};$$

$$\begin{aligned} \text{hence, } u' = \sin x + \cos x \frac{y}{1} - \sin x \frac{y^2}{1.2} - \cos x \frac{y^3}{1.2.3} \\ + \sin x \frac{y^4}{1.2.3.4} - \text{etc.} \end{aligned}$$

Making $x = 0$, whence $\sin x = 0$, and $\cos x = +1$, we have,

$$u' = \sin y = y - \frac{y^3}{1.2.3} + \frac{y^5}{1.2.3.4.5} + \text{etc., as already shown.}$$

2. $u' = l(x + y)$.

$$\text{Ans. } u' = lx + \frac{y}{x} - \frac{y^2}{2x^2} + \frac{y^3}{3x^3} - \frac{y^4}{4x^4} + \text{etc.}$$

3. $u' = a^{x+y}$.

$$\text{Ans. } u' = a^x \left(1 + (la)y + \frac{(la)^2}{1.2}y^2 + \frac{(la)^3}{1.2.3}y^3 + \text{etc.} \right).$$

4. $u' = \cos(x + y).$

$$\begin{aligned} \text{Ans. } u' = \cos x - \sin x \frac{y}{1} - \cos x \frac{y^2}{1.2} \\ + \sin x \frac{y^3}{1.2.3} + \cos x \frac{y^4}{1.2.3.4} - \text{etc.} \end{aligned}$$

V. DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES, AND OF IMPLICIT FUNCTIONS.

Geometrical Representation of a Function of two Variables.

23. It may be shown, as in Art. 3, that every function of two variables represents the ordinate of a surface of which those variables are the corresponding abscissas. If the vertical ordinate be taken as the function, it may be expressed by the equation,

$$z = f(x, y) \dots \dots (1)$$

In this equation, x and y are independent variables, and each may vary precisely as though the other were constant. To illustrate, let PQR be the surface whose equation is (1).

If, for a given value of y as OB , we suppose x to vary, equation (1) will represent a section of the surface parallel to the plane xz and at the distance OB from it. If for a given value of x , as OA , we suppose y to vary, equation (1) will represent a section parallel to the plane yz

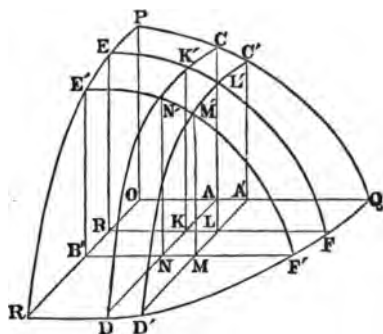


Fig. 2.

and at the distance OA from it. These sections will be called *parallel sections*, and the corresponding planes will be called *planes of parallel section*. The planes ACD and BEF determine the ordinate $KK' = z$. Let ACD , $A'C'D'$, be consecutive planes of parallel section, at a distance from each other equal to dx , and BEF , $B'E'F'$, also consecutive planes of parallel section, at a distance from each other equal to dy . These planes determine four ordinates KK' , LL' , NN' , and MM' , which may be denoted by the symbols z , z' , z'' , and z''' . Of these, z and z' are consecutive when x alone varies, z and z'' are consecutive when y alone varies, and z and z''' are consecutive when both x and y vary.

Differentials of a Function of two Variables.

24. The difference between two consecutive states of the function when x alone varies is called *the partial differential* with respect to x , and is denoted by the symbol $(dz)_x$; the difference between two consecutive states when y alone varies is called *the partial differential* with respect to y , and is denoted by the symbol $(dz)_y$; the difference between two consecutive states when both x and y vary is called the total differential, and is denoted by the ordinary symbol, dz without a subscript letter. Let us assume the figure and notation of Art. 23. We have, from the definition,

$z' = z + (dz)_x$; whence, by differentiation,

$$(dz')_y = (dz)_y + (d^2z)_{x,y}, \dots \dots (2)$$

The symbol $(d^2z)_{x,y}$, indicates the result obtained by differentiating z as though x were the only variable, and then

differentiating that result as though y were the only variable, the order of the subscript letters indicating the order of differentiation. From what precedes, we also have,

$$z''' = z' + (dz')_y;$$

substituting for z' and $(dz')_y$, their values taken from (2), we have,

$$z''' = z + (dz)_x + (dz)_y + (d^2z)_{x,y};$$

transposing z to the first member, replacing $z''' - z$ by its value dz , and neglecting the part $(d^2z)_{x,y}$, because it is an infinitesimal of the second order, we have, finally,

$$dz = (dz)_x + (dz)_y \dots \dots (3)$$

Hence, *the total differential is equal to the sum of the partial differentials.*

EXAMPLES.

1. $z = ax^3 + by^3.$

$$(dz)_x = 3ax^2dx; (dz)_y = 3by^2dy \therefore dz = 3ax^2dx + 3by^2dy.$$

2. $z = ax^2y^3.$

Ans. $dz = 2axy^3dx + 3ax^2y^2dy.$

3. $z = x^y.$

Ans. $dz = yx^{y-1}dx + x^y \ln x dy.$

Notation employed to designate Partial Differential Coefficients.

25. From the nature of the case, there can be no such thing as a differential coefficient of a function of two variables; but the quotient of a partial differential by the

differential of the corresponding variable, is called a *partial differential coefficient*. The form of the symbol indicates the variable with reference to which the function has been differentiated, and no subscript letter is required. Thus, in Example 2, Art. 24, we have,

$$\frac{dz}{dx} = 2ay^3x, \text{ and } \frac{dz}{dy} = 3ax^2y^2.$$

A similar notation is employed for partial differential coefficients of a higher order, as will be seen in the following article.

Successive Differentiation of Functions of two Variables.

26. The second differential of a function of two variables is found by differentiating the first differential by the rules already given. The third differential comes from the second in the same manner that the second comes from the first, and so on. In finding the higher partial differentials, the result obtained by differentiating the function first with respect to x , and that result with respect to y , is the same as though we had differentiated the function first with respect to y , and the result with respect to x . For, from Art. 24, we have,

$$z''' = z' + (dz')_y = z + (dz)_x + (dz)_y + (d^2z)_{x,y}.$$

And in like manner, we have,

$$z''' = z'' + (dz'')_x = z + (dz)_y + (dz)_x + (d^2z)_{y,x}.$$

Equating these values of z''' , and reducing, we find,

$$(d^2z)_{x,y} = (d^2z)_{y,x} \dots \dots (4)$$

which was to be shown. From equation (4), by an extension of the notation in Art. 25, we have,

$$\frac{d^2 z}{dx dy} = \frac{d^2 z}{dy dx}, \text{ or } \frac{d\left(\frac{dz}{dx}\right)}{dy} = \frac{d\left(\frac{dz}{dy}\right)}{dx} \dots\dots (5)$$

From what precedes, we have,

$$dz = (dz)_x + (dz)_y;$$

$$\begin{aligned} \therefore d^2 z &= (d^2 z)_{x,x} + (d^2 z)_{x,y} + (d^2 z)_{y,y} + (d^2 z)_{y,x} \\ &= (d^2 z)_{x,x} + 2(d^2 z)_{x,y} + (d^2 z)_{y,y}. \end{aligned}$$

EXAMPLES.

1. Given, $z = x^3 y^3$, to find $d^2 z$.

$$\begin{aligned} (d^2 z)_{x,x} &= 6y^3 x dx^2; \quad (d^2 z)_{x,y} = 6x^2 y dx dy; \\ (d^2 z)_{y,y} &= 2x^3 dy^2. \end{aligned}$$

$$\therefore d^2 z = 6y^3 x dx^2 + 12x^2 y dx dy + 2x^3 dy^2.$$

2. Given, $z = y^3 x^{\frac{2}{3}}$, to find $d^2 z$.

$$\text{Ans. } d^2 z = \frac{15}{4} y^3 x^{\frac{1}{3}} dx^2 + 2\left(\frac{15}{2} y^3 x^{\frac{2}{3}}\right) dy dx + 6x^{\frac{2}{3}} y dy^2$$

8

Extension to three or more Variables,

27. If we have a function of three or more independent variables, we may find its differential by differentiating separately with respect to each variable, and taking the sum of the partial differentials thus obtained. The second and higher differentials are found in an entirely analogous manner.

Definition of Explicit and Implicit Functions.

28. An *explicit* function, is one in which the value of the function is directly expressed in terms of the variable. Thus, $y = \sqrt{2rx - x^2}$, is an explicit function. An *implicit* function, is one in which the value of the function is not directly given in terms of the variable. Thus, in the equation,

$$ay^2 + bxy + cx^2 + d = 0;$$

y is an implicit function of x . Implicit functions are generally connected with their variables by one or more equations. When these equations are solved the implicit function becomes explicit.

Differentials of Implicit Functions.

29. The differential of an implicit function may be found without first finding the function itself. For, if we differentiate both members of the first equation, we shall thus find a new equation, which, with the given one, will enable us to find either the differential, or the differential coefficient of the function. For example, suppose we have the equation,

$$y^2 + 2xy + x^2 - a^2 = 0 \dots \dots (1)$$

to find the differential coefficient of y . Differentiating (1), and dividing by 2, we have,

$$ydy + xdy + ydx + xdx = 0 \dots\dots (2)$$

Finding the value of $\frac{dy}{dx}$, we have,

$$\frac{dy}{dx} = -1.$$

Again, let us have the relation,

$$xy = m \dots\dots (1)$$

to find the value of $\frac{dy}{dx}$. Differentiating (1), we have,

$$xdy + ydx = 0 \dots\dots (2)$$

Whence,

$$\frac{dy}{dx} = -\frac{y}{x} \dots\dots (3)$$

But from (1), we have,

$$\frac{y}{x} = \frac{m}{x^2}.$$

Substituting in (3), we find,

$$\frac{dy}{dx} = -\frac{m}{x^2} \dots\dots (4)$$

EXAMPLES.

Find the first and second differential coefficients of y in the following implicit functions:

1. $y^3 - 3y + x = 0.$

We have,

$$3y^2 dy - 3dy + dx = 0, \therefore \frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{1-y^2};$$

also,

$$6y dy^2 + 3y^2 d^2 y - 3d^2 y = 0,$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{2y}{1-y^2} \cdot \frac{dy^2}{dx^2} = \frac{2}{9} \cdot \frac{y}{(1-y^2)^3}.$$

2. $y^2 - 2mxy + x^2 - a = 0.$

$$\text{Ans. } \frac{dy}{dx} = \frac{my - x}{y - mx}, \text{ and } \frac{d^2 y}{dx^2} = \frac{a(m^2 - 1)}{(y - mx)^3}.$$

PART II.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS.

I. TANGENTS AND ASYMPTOTES.

Geometrical Representation of the First Differential Coefficient.

30. It has been shown (Art. 3), that any function of one variable may be represented by the ordinate of a curve, of which that variable is the corresponding abscissa. We have also seen (Art. 5), that an element of the curve, PQ , does not differ from a straight line. This line, prolonged toward T , is tangent to the curve at P . The angle that the tangent makes with the axis of X is equal to RPQ ; denoting it by θ , and remembering that the tangent of the angle at the base of a right angled triangle is equal to the perpendicular divided by the base, we have,

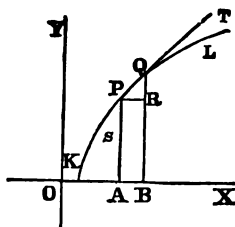


Fig. 3.

$$\tan \theta = \frac{dy}{dx}.$$

The tangent of θ is taken as the measure of the *slope*, not only of the tangent, but also of the curve at the point P . Hence, *the slope of a curve, at any point, is measured*

by the first differential coefficient of the ordinate at that point.

The slope is positive or negative, that is, the curve slopes upward or downward, according as the first differential coefficient is *plus*, or *minus*.

Applications.

31. The principle just demonstrated enables us to find the point of a curve, at which the tangent makes a given angle with the axis of x .

Thus, let it be required to find the point on a given parabola at which the tangent makes an angle of 45° with the axis. Assume the equation of the parabola,

$$y^2 = 2px; \therefore \frac{dy}{dx} = \frac{p}{y}.$$

Placing this equal to 1, we have,

$$\frac{p}{y} = 1, \text{ or } y = p.$$

But $y = p$, is the ordinate through the focus. Hence, the required point is at the upper extremity of the ordinate through the focus.

Again, let it be required to find the point at which a tangent to the ellipse is parallel to the axis of x . Assume the equation,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1; \therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

Placing this equal to 0, we have,

$$-\frac{b^2x}{a^2y} = 0,$$

which can only be satisfied by making $x = 0$; this, in the equation of the curve, gives $y = \pm b$. Hence, the tangent at either vertex of the conjugate axis fulfills the given condition.

Again, to find where the tangent to an hyperbola is perpendicular to the axis; assume the equation of the curve,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \dots\dots (1)$$

whence, by the rule,

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \infty; \therefore y = 0, x = \pm a.$$

Hence, the tangent at either extremity of the transverse axis, fulfills the given condition.

Equations of the Tangent and Normal.

32. Let P be a point of the curve whose co-ordinates are x'', y'' ; then will the equation of a straight line through it, be of the form,

$$y - y'' = a(x - x'') \dots\dots (1)$$

in which a is the slope. If we make a equal the tangent of RPQ , the line will be tangent to the curve. But the tangent of RPQ is $\frac{dy''}{dx''}$; hence, we have, for the equation of a tangent line,

$$y - y'' = \frac{dy''}{dx''}(x - x'') \dots\dots (2)$$

If we make a equal to *minus* the reciprocal of the tangent of RPQ , the line will be perpendicular to the tangent,

that is, it will be normal to the curve; hence, we have, for the equation of a normal line,

$$y - y'' = -\frac{dx''}{dy''}(x - x'') \dots \dots (3)$$

In these equations x and y are general co-ordinates of the lines, and x'' , y'' are the co-ordinates of the point of contact.

Let it be required to find the equation of a tangent to an ellipse. We found, in the last article, the value of $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$; substituting for x and y the particular values x'' and y'' , and putting the result in (2), we find,

$$y - y'' = -\frac{b^2x''}{a^2y''}(x - x'');$$

whence, by reduction,

$$\frac{yy''}{b^2} + \frac{xx''}{a^2} = 1 \dots \dots (4)$$

Making $a = b$, in this equation, we have,

$$yy'' + xx'' = a^2,$$

which is the equation of a tangent to a circle.

To find the equation of a normal to an ellipse we substitute the value of $\frac{dx''}{dy''}$, in equation (3), which gives,

$$y - y'' = \frac{a^2y''}{b^2x''}(x - x'') \dots \dots (5)$$

Making $a = b$, we have,

$$y - y'' = \frac{y''}{x''}(x - x''),$$

which is the equation of a normal to a circle.

If, in equation (2), we make $y = 0$, the corresponding value of $x - x''$ will express the length of the subtangent, counted from the foot of the ordinate through the point of contact; denoting this by $S.T.$, we have,

$$S.T = -y'' \frac{dx''}{dy''} \dots \dots (6)$$

If, in equation (3), we make $y = 0$, the corresponding value of $x - x''$ will express the length of the subnormal, counted from the foot of the ordinate through the point of contact; denoting this by $S.N.$, we have,

$$S.N = y'' \frac{dy''}{dx''} \dots \dots (7)$$

Substituting in (6) and (7) the values of $\frac{dy''}{dx''}$ taken from the equation of the ellipse, we have,

$$S.T = + \frac{a^2 y''^2}{b^2 x''}, \text{ and } S.N = - \frac{b^2 x''}{a^2}.$$

Asymptotes.

33. An *asymptote* to a curve, is a line that continually approaches the curve and becomes tangent to it at an infinite distance. The characteristics of an asymptote are, that it is tangent to the curve at a point that is infinitely distant, and that it cuts one or both axes at a finite distance from the origin.

To ascertain whether a curve has an asymptote, assume the equation of the tangent (Art. 32), and in it make x , and y , successively equal to 0.

In this manner we find,

$$AD = y'' - x' \frac{dy''}{dx''}, \text{ and}$$

$$AT = x'' - y' \frac{dx''}{dy'} \dots \dots (1)$$

We then find $\frac{dy''}{dx''}$ from the equation of the curve, substitute it in (1) and make the hypothesis that places the point (x'', y'') at an infinite distance. If this supposition make either AD or AT finite, the curve has an asymptote, otherwise not.

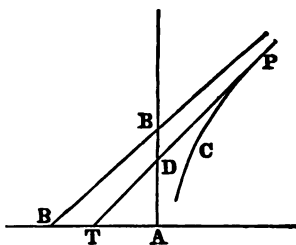


Fig. 4.

Let it be required to find whether the hyperbola has an asymptote. We have found (Art. 31), $\frac{dy''}{dx''} = \frac{b^2 x''}{a^2 y''}$, which in (1) gives,

$$AD = y'' - \frac{b^2 x''^2}{a^2 y''}, \text{ and } AT = x'' - \frac{a^2 y''^2}{b^2 x''} \dots \dots (2)$$

Whence, by reduction,

$$AD = \frac{a^2 y''^2 - b^2 x''^2}{a^2 y''}, \text{ and } AT = \frac{b^2 x''^2 - a^2 y''^2}{b^2 x''} \dots \dots (3)$$

or,
$$AD = -\frac{b^2}{y''}, \text{ and } AT = \frac{a^2}{x''} \dots \dots (4)$$

The only hypothesis that places a point of the hyperbola at an infinite distance is $y'' = \pm \infty$, and $x'' = \pm \infty$; both these sets of values, in (4), make AD and AT equal to 0. Hence, the hyperbola has two asymptotes, both passing through the centre.

To find whether the parabola has an asymptote, make $\frac{dy'}{dx''} = \frac{p}{y''}$ in (1), whence,

$$AD = y'' - \frac{px''}{y''} = \frac{px''}{y''}, \text{ and } AT = -x''.$$

If we make $y'' = \infty$, and $x'' = \infty$, to put the point of contact at an infinite distance, we find both AD and AT equal to ∞ , which shows that the parabola has no asymptote.

Order of Contact.

34. Two lines have a contact of the *first order*, when they have two consecutive points in common. The first of these is *the point of contact*; this point is common to both lines, and, as we have just seen, the first differential coefficients of the ordinates of the two lines at this point are equal. Conversely, if two lines have a common point, and if the first differential coefficients of the ordinates of the lines at that point are equal, the lines have a contact of the first order.

Two lines have a contact of the *second order*, when they have three consecutive points in common. The first of the three is *the point of contact*, and because the lines have three consecutive ordinates common, the first and second differential coefficients of their ordinates at this point are equal. Conversely, if two lines have a common point, and the first and second differential coefficients of their ordinates at that point equal, the lines have a contact of the *second order*.

In like manner it may be shown, if two lines have a point in common, and n successive differential coefficients of their ordinates at that point equal, that the lines have

a contact of the n^{th} order, n being any positive whole number.

If a line be applied so as to cut a given line in an *even* number of points, it will, just before the first, and just after the last, lie on the same side of the given line; and this is true when the points of secancy are consecutive, in which case the lines have a contact of an *odd* order. Hence, if two lines have a contact of an odd order, they do *not intersect* at the point of contact. If the applied line cut the given line in an *odd* number of points, it will, just before the first, and just after the last, lie on opposite sides of the given line; and this is also true when the points of secancy are consecutive; hence, if two lines have a contact of an even order, they *intersect* at the point of contact.

If the applied line be straight, and the contact of the n^{th} order, the given curve will have n consecutive values of dy , equal to each other. If each of these be taken from the next in order, the differences will be 0, that is, the curve will have $n - 1$ consecutive values of d^2y equal to 0. In like manner, we may show that it has $n - 2$ consecutive values of d^3y equal to 0, and so on, to the $(n + 1)^{\text{th}}$ differential of y , which will not be 0, but will be *plus* or *minus*, according as the $(n + 1)^{\text{th}}$ value of dy , counting from the point of contact, is *greater* or *less* than the n^{th} value of dy . Conversely, if the $(n + 1)^{\text{th}}$ differential of y is *plus*, the $(n + 1)^{\text{th}}$ value of dy is greater than the n^{th} ; if *minus* the $(n + 1)^{\text{th}}$ value of dy is less than the n^{th} .

II. CURVATURE.

Direction of Curvature.

35. Let KL be a curve, whose concavity is turned downward, that is, in the direction of negative ordi-

nates ; let AP , BQ , and CQ' be consecutive ordinates, AB and BC being equal to dx ; prolong PQ toward T . Then will RQ , and $R'Q'$, be consecutive values of dy , and consequently their difference, $R'Q' - RQ$, will be the value of d^2y at P . The right angled triangles, PRQ , and $QR'S$, have their bases, and the angles at their bases, equal ; hence, their altitudes, RQ , and $R'S$, are equal ; but $R'Q'$ is less than $R'S$, because Q' is nearer ∞ than S ; hence, $R'Q'$ is less than RQ , and consequently the value of d^2y is *negative*. Conversely, if d^2y is *negative*, $R'Q'$ is

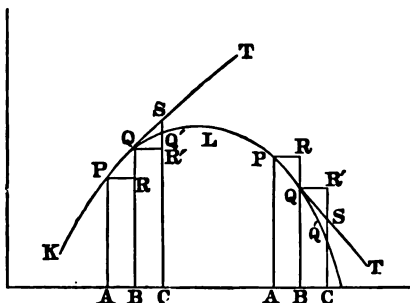


Fig. 5.

less than $R'S$, and the curve in passing from P is *concave downward*. In like manner, it may be shown that the curve is *concave upward* at P , when d^2y is *positive*. Because the sign of the second differential coefficient is the same as that of the second differential, we have the following rule for determining the direction of curvature, in passing from any point toward the right :

Find the second differential coefficient of the ordinate at the point ; if this is negative, the curve is concave downward, if positive, it is concave upward.

We may regard y as the independent variable, and find the second differential coefficient of x ; if this be negative at any point, the concavity is turned toward the left, if positive, toward the right.

Example.—Let it be required to determine the direction of curvature at any point of a parabola.

Assume the equation, $y^2 = 2px$; whence, by differentiation and reduction,

$$\frac{d^2y}{dx^2} = -\frac{p^2}{y^3}, \text{ and } \frac{d^2x}{dy^2} = \frac{1}{p}.$$

The *first* of these is negative for positive values of y , and positive for negative values of y ; hence, the part of the curve above the axis of X is concave downward, and the part below is concave upward. The *second* is positive for all values of x and y ; hence, the curve is everywhere concave toward the right.

If the tangent at P have a contact of the n^{th} order, n being greater than 1, the second differential coefficient of y at that point will be 0 (Art. 34). In this case it may be shown, as above, that the curve bends downward in passing toward the right, when the $(n+1)^{\text{th}}$ value of dy , counting from P , is less than the n^{th} value of dy , and upward when the $(n+1)^{\text{th}}$ value of dy is greater than the n^{th} . The former condition is satisfied when the $(n+1)^{\text{th}}$ differential coefficient of y is *negative*, and the latter when it is *positive* (Art. 34).

Amount of Curvature.

36. The *change of direction* of a curve in passing from an element to the one next in order, is the angle between the second element and the prolongation of the first. This angle is called the *angle of contingence*, it is *negative* when the curve bends downward, and *positive* when it bends upward. The total change of direction in passing over any arc is equal to the algebraic sum of the angles of contingence at every point of that arc.

The *curvature* of a curve is the *rate* at which it changes direction. When the curvature is uniform, as in the

circle, it is measured by the total change of direction in any arc, divided by the length of that arc. In any curve whatever the curvature may be regarded as uniform for an infinitely small arc; hence, the curvature at any point of a curve, is equal to the angle of contingence at that point, divided by the corresponding element of the curve. Denoting the angle of contingence by $d\theta$, the corresponding element by ds , and the curvature by c , we have,

$$c = \frac{d\theta}{ds} \dots \dots (1)$$

Osculatory Circle.

37. An *osculatory circle*, to a given curve, is a circle that has three consecutive points in common with that curve. Thus, if the circle whose centre is C , pass through the three consecutive points P , Q , and R , of the curve, KQ , it is osculatory to that curve at P .

The circle and curve have two consecutive tangents, PT' and QT'' , and two consecutive normals, PC and QC , in common; they have also the angle of contingence at Q , in common; hence, they have the same curvature at the point P . Because PC and QC are perpendicular to PT , and QT' , their included angle, is equal to

The diagram shows a circle and a curve passing through points P and Q. At point P, a horizontal line segment PC is perpendicular to a tangent line PT. At point Q, a horizontal line segment QC is perpendicular to a tangent line QT'. The angle between the normals PC and QC is labeled as the angle of contingence. Other points and lines are labeled: A, O, E, B, D, H, K, R, T, T'.

Fig. 6.

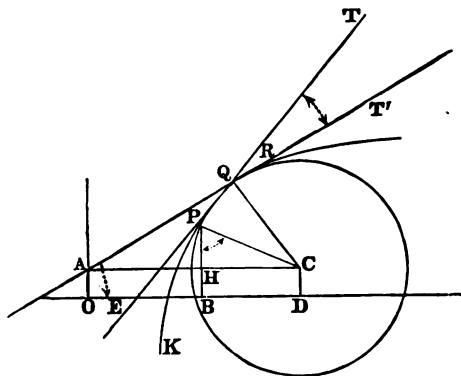


Fig. 6.

the angle of contingence, $d\theta$; hence, $PQ = CP \times d\theta$, or denoting PQ by ds and CP by R , we have, $ds = R d\theta$, or,

$$\frac{d\theta}{ds} = \frac{1}{R} \dots \dots (1)$$

This equation shows that the curvature of a curve at any point is equal to the reciprocal of the radius of the osculatory circle at that point; for this reason, the radius of the osculatory circle is called the *radius of curvature*.

Radius of Curvature.

38. If we denote the inclination of the tangent, PT , to the axis of X by θ , the angle of contingence, TQT' , equal to PCQ , will be equal to $d\theta$; but we know that $\theta = \tan^{-1} \frac{dy}{dx}$; differentiating, we have,

$$d\theta = \frac{\frac{d^2y}{dx^2}}{1 + \frac{dy^2}{dx^2}} = \frac{dxd^2y}{dy^2 + dx^2}.$$

Finding the value of R from equation (1), Art. 37, and replacing ds by its value, from Art. 5, we have,

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} \dots \dots (1)$$

Dividing both terms of the second member by dx^3 , and denoting the first differential coefficient of y by q' , and its second differential coefficient by q'' , we have,

$$R = \frac{(1 + q'^2)^{\frac{3}{2}}}{q''} \dots \dots (2)$$

When the curve bends downward, $d\theta$ is negative, and consequently R is *negative*; when the curve bends upward, both are *positive*.

It is sometimes convenient to regard R , or some other quantity on which R depends, as the independent variable. In this case both dy and dx are variable, and we have,

$$d\theta = \frac{dx d^2y - dy d^2x}{dx^2 + dy^2};$$

and this in (1), Art. 37, gives,

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx d^2y - dy d^2x} \dots \dots (3)$$

Example.—Required the value of R for the parabola. Assume the equation, $y^2 = 2px$.

By differentiation and reduction, we have,

$$\frac{dy}{dx} = q' = \frac{p}{y}, \text{ and } \frac{d^2y}{dx^2} = q'' = -\frac{p^2}{y^3};$$

substituting these in (2), we have,

$$R = \left(1 + \frac{p^2}{y^2}\right)^{\frac{3}{2}} \times -\frac{y^3}{p^2} = -\frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2}.$$

The value of R is least possible when $y = 0$, increases as y increases, and becomes ∞ when $y = \infty$. Hence, the curvature of the parabola is greatest at the vertex, and diminishes as the curve recedes from the vertex.

Co-ordinates of the Centre of the Osculatory Circle.

39. In the right angled triangle, PHC , PH is perpendicular to ED , and PC to EP ; hence, the angle, HPC , is

equal to θ . Denoting the abscissa, AC , by α , and the ordinate, DC , by β , we have,

$$\alpha = AH + HC, \text{ and } \beta = BP - HP \dots \dots (1)$$

AH and BP are the co-ordinates of P , denoted by x and y ; HP is equal to $PC \cos \theta$, and HC to $PC \sin \theta$; substituting for PC , or R , its value found in the last article, remembering that it is *negative* in the case under consideration, (Art. 38), replacing $\sin \theta$, and $\cos \theta$, by their values taken from Art. 5, and reducing, we have,

$$\left. \begin{aligned} \alpha &= x - \frac{dx^2 + dy^2}{d^2y} \cdot \frac{dy}{dx} = x - \left(\frac{1 + q'^2}{q''} \right) q' \\ \beta &= y + \frac{dx^2 + dy^2}{d^2y} = y + \frac{1 + q'^2}{q''} \end{aligned} \right\} \dots \dots (2)$$

Locus of the Centre of Curvature.

40. A line drawn through the centres of the osculatory circles at every point of a curve is called the *evolute* of that curve, and the given curve, with respect to its evolute, is called an *involute*. If we consider any radius of curvature of a curve, we see that it intersects the one that precedes and the one that follows it, and the distance between the points of intersection is an element of the evolute; hence, the radius is normal to the involute, and tangent to the evolute. We also see that the difference between two consecutive radii of curvature is equal to the corresponding element of the evolute; hence, the difference between any two radii of curvature of a given curve is equal to the corresponding arc of the evolute.

The evolute of a curve may be constructed by drawing normals to the curve, and then drawing a curve tangent

to them all: the closer the normals, the more accurate will be the construction. An involute may be constructed by wrapping a thread round the evolute, holding it tense, and then unwrapping it. Every point of the thread describes a curve, which is an involute of the given evolute. Practically, the evolute is cut out of a board, or other solid material. In this manner the outlines of the teeth of wheels are sometimes marked out, the circle being taken as an evolute.

Equation of the Evolute of a Curve.

41. The equation of the evolute of any curve may be found by combining the equation of the curve with formulas (2), Art. 39, and eliminating x and y . The method will be best illustrated by an example; thus, let it be required to find the equation of the evolute of the common parabola.

We have already found $q' = \frac{p}{y}$, and $q'' = -\frac{p^2}{y^3}$ (Art. 38); substituting these in (2), Art. 39, we have, after reduction,

$$\alpha = x + \frac{y^2 + p^2}{p} = 3x + p, \therefore x = \frac{1}{3}(\alpha - p);$$

$$\beta = y - \frac{y(y^2 + p^2)}{p^2} = -\frac{y^3}{p^2}, \therefore y = -\beta^{\frac{1}{3}}p^{\frac{2}{3}};$$

substituting the values of x and y in the equation $y^2 = 2px$, we have,

$$\beta^{\frac{2}{3}}p^{\frac{4}{3}} = \frac{2p}{3}(\alpha - p), \therefore \beta^2 = \frac{8}{27p}(\alpha - p)^3,$$

which is the equation required. It is the equation of a semi-cubic parabola.

EXAMPLES.

1. Find the equation of the evolute of an ellipse.

$$\text{Ans. } (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

- X 2. Find the equation of the evolute of an hyperbola.

$$\text{Ans. } (ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$$

3. Find the radius of curvature of an ellipse.

$$\text{Ans. } \rho = - \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}.$$

4. Find the radius of curvature of an equilateral hyperbola referred to its asymptotes, the equation being
- $xy = m$
- .

$$\text{Ans. } \rho = \frac{(m^2 + x^4)^{\frac{3}{2}}}{2mx^3}.$$

III. SINGULAR POINTS OF CURVES.

Definition of a Singular Point.

42. A SINGULAR POINT of a curve, is a point at which the curve presents some peculiarity not common to other points. The most remarkable of these are, *points of inflexion, cusps, multiple points, and conjugate points.*

Points of Inflexion.

43. A POINT OF INFLEXION is a point at which the curvature changes, from being concave downward to being concave upward, or the reverse.

Inasmuch as the direction of curvature is determined by the sign of the second differential coefficient of the ordi-

nate, it follows that this sign must change from $-$ to $+$, or from $+$ to $-$, in passing a point of inflexion. But a quantity can only change sign by passing through 0, or ∞ . Hence, at a point of inflexion the second differential coefficient of y must be either 0, or ∞ . If the corresponding values of x and y are such, that for values immediately preceding and following them, the second differential coefficient has contrary signs, then will each set of such values correspond to a point of inflexion.

EXAMPLES.

1. To find the points of inflexion on the curve whose equation is $y = b + (x - a)^3$.

We find, $\frac{d^2y}{dx^2} = 6(x - a)$; this, put equal to 0, gives $x = a$, which, in the equation of the curve, gives $y = b$. When $x < a$, the second differential coefficient is *negative*, and when $x > a$, it is *positive*. Hence, the point whose co-ordinates are a and b , is a point of inflexion.

2. To find the points of inflexion on the curve whose equation is $y = b + \frac{1}{4}(x - a)^5$.

We find, $\frac{d^2y}{dx^2} = 2(x - a)^3$; this placed equal to 0, gives $x = a$, whence, $y = b$. When $x < a$, the second differential coefficient is *negative*, and when $x > a$, it is *positive*. Hence, the point whose co-ordinates are a and b , is a point of inflexion.

3. Find the co-ordinates of the points of inflexion on

the curve whose equation is $y = 2r\sqrt{\frac{2r - x}{x}}$.

$$\text{Ans. } x = \frac{3}{2}r, \text{ and } y = \frac{2}{3}r\sqrt{3}.$$

4. Find the co-ordinates of the points of inflexion on the line whose equation is $y = \frac{x^2(a^2 - x^2)}{a^3}$.

$$\text{Ans. } x = \pm \frac{1}{6}a\sqrt{6}, \text{ and } y = \frac{5}{36}a.$$

Cusps.

44. A CUSP is a point at which two branches terminate, being tangential to each other. There are two species of cusps; the *ceratoid*, named from its resemblance to the horns of an animal, and the *rampoid*, named from its resemblance to the beak of a bird. The first is shown at *E*, and the second at *A*.

The method of determining the position and nature of a cusp point will be best shown by examples.

1. Let us take the curve whose equation is $y = b \pm (x - a)^{\frac{3}{2}}$. From this we find,

$$\frac{dy}{dx} = \pm \frac{3}{2}(x - a)^{\frac{1}{2}}, \text{ and } \frac{d^2y}{dx^2} = \pm \frac{3}{4}(x - a)^{-\frac{1}{2}}.$$

For $x = a$, $\frac{dy}{dx} = 0$, and $\frac{d^2y}{dx^2} = \infty$. For $x < a$, both are imaginary, as is also the value of y . For $x > a$, both are real, and each has two values, one *plus* and the other *minus*. Hence, the point whose co-ordinates are a and b is a cusp of the first species.

2. Let us take the curve whose equation is $y = x^2 \pm x^{\frac{5}{2}}$. From it we find,

$$\frac{dy}{dx} = 2x \pm \frac{5}{2}x^{\frac{3}{2}}, \text{ and } \frac{d^2y}{dx^2} = 2 \pm \frac{15}{4}x^{\frac{1}{2}}.$$

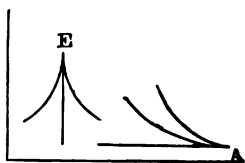


Fig. 7.

From the given equation we see that no part of the curve lies to the left of the axis of y , and that there are two infinite branches to the right of that axis.

The values of $\frac{dy}{dx}$ are both 0 at the origin; and at that point both values of $\frac{d^2y}{dx^2}$ are positive; hence, the origin is a cusp of the second species.

A discussion of the above equations shows that the upper branch has its concavity always upward; the second branch has a point of inflexion for, $x = \frac{64}{225}$; its slope is 0

for $x = \frac{16}{25}$; and it cuts the axis of x at the distance 1 from the origin; from the origin to the point of inflexion, it curves upward; after that point it curves downward. The shape of the curve is indicated in the figure.

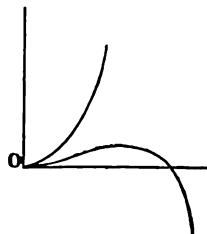


Fig. 8.

For the *ceratoid* the values of the second differential coefficient of y have contrary signs, for the *ramphoid* they have the same sign.

Multiple Points.

45. A MULTIPLE POINT is a point where two or more branches of a curve intersect, or touch each other.

If the branches intersect, $\frac{dy}{dx}$ will have as many values at that point as there are branches; if they are tangent, these values will be equal.

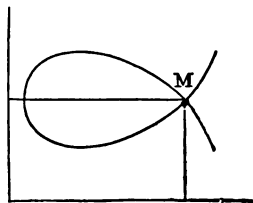


Fig. 9.

EXAMPLES.

1. Take the curve whose equation is $y^3 = x^3 - x^4$.

For every value of x there are two values of y equal with contrary signs; hence, the curve is symmetrically situated with respect to the axis of x . For $x = 0$, we have $y = \pm 0$; hence, both branches pass through the origin. From the equation, we find,

$$\frac{dy}{dx} = \frac{x}{y} - \frac{2x^3}{y} = \pm \frac{1}{\sqrt{1-x^3}} \mp \frac{2x^3}{\sqrt{1-x^3}}.$$

For $x = 0$, $\frac{dy}{dx} = \pm 1$, hence the origin is a multiple point, by intersection. The curve is limited in both directions.

2. Take the curve whose equation is $y = \pm \sqrt{x^5 + x^4}$.

The two branches are symmetrically placed with respect to the axis of x , both pass through the origin, forming a loop on the left and extending to infinity on the right. We find, from the equation of the curve,

$$\frac{dy}{dx} = \pm \frac{5x^4 + 4x^3}{2\sqrt{x^5 + x^4}} = \pm \frac{5x^2 + 4x}{2\sqrt{x+1}}$$

For $x = 0$, we have $\frac{dy}{dx} = \pm 0$; hence, the origin is a multiple point, the branches being tangent to each other and to the axis of x .

3. The curve whose equation is $y = \pm x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

has a multiple point at the origin, and is composed of two pointed loops, one on the right and the other on the left of the origin.

4. The curve whose equation is $y^2 = (x - 2)(x - 3)^2$ is symmetrical with respect to the axis of x and consists of a looped branch which extends from $x = 2$ to $x = 3$; at the latter point the upper part passes below the axis of x and lower part passes above that axis, forming a multiple point; beyond $x = 3$ the two parts continually diverge, extending to $x = \infty$.

Conjugate Points.

46. A CONJUGATE, OR ISOLATED POINT, is a point whose co-ordinates satisfy the equation of a curve, but which has no consecutive point. Because it has no consecutive point, the value of the first differential coefficient of y at it, is imaginary.

EXAMPLES.

1. Take the curve whose equation is $y = \pm x \sqrt{x - a}$.

In this case $x = 0$ and $y = 0$, satisfy the equation, but no other value of x less than a gives a point of the curve. Furthermore, we have,

$$\frac{dy}{dx} = \pm \frac{1}{2} \frac{3x^2 - 2ax}{\sqrt{x^3 - ax^2}},$$

which for $x = 0$ is imaginary, as also for all values of $x < a$.

2. Take the curve whose equation is

$$y = ax^2 \pm \sqrt{x(1 - \cos x)}.$$

For every positive value of x , there are two values of y , and consequently two points, except when $\cos x = 1$, when the two points reduce to one. These points constitute a series of loops like the links of a chain, having for a dia-

metral curve a parabola whose equation is $y = ax^2$. For every negative value of x the values of y are imaginary, except when $\cos x = 1$; in these cases we have a series of isolated points situated on the diametral curve, whose equation is $y = ax^2$. We have, by differentiating and reducing,

$$\frac{dy}{dx} = 2ax \pm \frac{\sqrt{1 - \cos x} + x\sqrt{1 + \cos x}}{2\sqrt{x}},$$

which is imaginary for negative values of x .

IV. MAXIMA AND MINIMA.

Definitions of Maximum and Minimum.

47. A function of one variable is at a *maximum state*, when it is *greater* than the states that immediately precede and follow it; it is at a *minimum state*, when it is *less* than the states that immediately precede and follow it. Thus, if KL be the curve of the function,

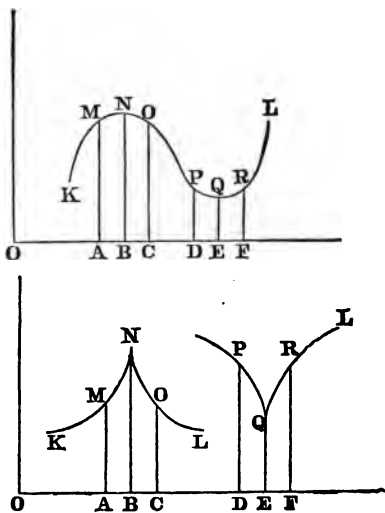


Fig. 10.

FR.

Analytical Characteristics of a Maximum or Minimum.

48. If we examine the figures given in the last article, we see that the slope of the curve KL , or the differential coefficient of the function, changes from $+$ to $-$ in passing a maximum, and from $-$ to $+$ in passing a minimum. But a varying quantity can only change sign by passing through 0, or ∞ ; hence, the first differential coefficient of the function must be either 0, or ∞ , at either maximum or minimum state.

If, therefore, we find the first differential coefficient of the function, and set it equal to 0 and ∞ , we shall have two equations which will give all the values of the variable that belong to either a maximum or minimum state of the function.

They may also give other values; hence the necessity of testing each root separately. This may be done by the following rule:

Subtract from, and add to, the root to be tested, an infinitely small quantity; substitute these successively in the first differential coefficient; if the first result is plus and the second minus, the root corresponds to a maximum; if the first is minus and the second plus, it corresponds to a minimum; if both have the same sign, it corresponds to neither a maximum nor a minimum.

The maximum or minimum value may be found by substituting the corresponding root in the given function.

EXAMPLES.

1. Let $y = 3 + (x - 2)^2$.

By the rule, we have, $\frac{dy}{dx} = 2(x - 2) = 0$, $\therefore x = 2$.

Substituting for x the values, $2 - dx$, and $2 + dx$, we find

for the corresponding values of the differential coefficient $-dx$, and $+dx$; hence, $x = 2$, corresponds to a *minimum*, which is $y = 3$.

2. Let $y = 4 - (x - 3)^{\frac{2}{3}}$.

We have,

$$\frac{dy}{dx} = -\frac{2}{3}(x-3)^{-\frac{1}{3}} = \infty; \therefore x = 3.$$

For $x = 3 - dx$, and $3 + dx$, we have the differential coefficient equal to

$$\frac{2}{3\sqrt[3]{dx}}, \text{ and } -\frac{2}{3\sqrt[3]{dx}},$$

hence, $x = 3$ corresponds to a *maximum*, $y = 4$.

3. Let $y = 3 + 2(x - 1)^3$.

We have,

$$\frac{dy}{dx} = 6(x-1)^2 = 0; \therefore x = 1.$$

Substituting $1 - dx$, and $1 + dx$, for x , in the first differential coefficient, we have in both cases a positive result; hence, $x = 1$ does not correspond to either a maximum, or minimum.

The preceding method is applicable in all cases; but, when the first differential coefficient of the function is 0, as it is in most instances, there is an easier process for testing the roots. In this case, the function being represented by the ordinate of a curve, the maximum and minimum states correspond to points at which the tangent is horizontal; furthermore, the curve lies wholly *above*, or wholly *below*, the tangent at the point of contact, that is, the tangent does not cut the curve at that point; hence, the contact is of an odd order, and consequently *the first of the successive differential coefficients of the function that*

does not reduce to 0 must be of an even order (Art. 34). If this is *negative*, the curve bends downward after passing the point of contact, and the root corresponds to a *maximum*, if *positive* the curve bends upward, and the root corresponds to a *minimum* (Art. 35). Hence the following practical rule for finding the values of the variables that correspond to maxima and minima:

Place the first differential coefficient of the function equal to 0, and solve the resulting equation; substitute each root in the successive differential coefficients of the function, until one is found that does not reduce to 0; if this is of an even order and negative, the root corresponds to a maximum, if of an even order and positive, to a minimum; but if of an odd order, it corresponds to neither maximum nor minimum.

EXAMPLES.

1. Let $y = x^3 - 3x + 2$.

We have,

$$\frac{dy}{dx} = 2x - 3 = 0; \quad \therefore x = \frac{3}{2}.$$

Also,

$$\frac{d^2y}{dx^2} = 2; \quad \therefore \left[\frac{d^2y}{dx^2} \right]_{\frac{3}{2}} = +2.$$

Hence, $x = \frac{3}{2}$, corresponds to a minimum state, which is,

$$y = [x^3 - 3x + 2]_{\frac{3}{2}} = -\frac{1}{4}.$$

The symbol $\left[\frac{d^2y}{dx^2} \right]_{\frac{3}{2}}$ is used to denote what $\frac{d^2y}{dx^2}$ becomes

when the variable that enters it is made equal to $\frac{3}{2}$.

2. $y = 4 - (x - 3)^4.$

$$\frac{dy}{dx} = -4(x - 3)^3 = 0; \therefore x = 3;$$

$$\frac{d^2y}{dx^2} = -12(x - 3)^2; \therefore \left[\frac{d^2y}{dx^2}\right]_3 = 0;$$

we also find,

$$\left[\frac{d^3y}{dx^3}\right]_3 = 0; \text{ and } \left[\frac{d^4y}{dx^4}\right]_3 = -24.$$

Hence, $[y]_3 = 4$, is a maximum.

In applying the above rule, a *positive constant factor* may be suppressed at any stage of the process; for it is obvious that such suppression can in no way alter the *value* of the roots to be found, or the *signs* of the successive differential coefficients.

3. Let $y = 3x^3 - 9x^2 - 27x + 30.$

Rejecting the factor, + 3, and denoting the result by u , we have,

$$\frac{du}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 0; \therefore x = 3, x = -1.$$

Rejecting the factor, + 3, and denoting the resulting value of the function by u' , we have,

$$\left[\frac{d^2u'}{dx^2}\right]_3 = 4, \text{ and } \left[\frac{d^2u'}{dx^2}\right]_{-1} = -4.$$

The first corresponds to a minimum and the second to a maximum. Substituting in the given function, we have, for the maximum and minimum values,

$$y' = 45, \text{ and } y'' = -51.$$

When the first differential coefficient is composed of

variable factors, the method of finding the corresponding values of the second differential coefficient may be simplified as follows: let us have,

$$\frac{dy}{dx} = P \times Q;$$

P and Q being functions of x , and suppose that $[P]_a = 0$. By the rule for differentiating, we have,

$$\frac{d^2y}{dx^2} = P \frac{dQ}{dx} + Q \frac{dP}{dx};$$

but because P becomes 0 when $x = a$, we have,

$$\left[\frac{d^2y}{dx^2} \right]_a = \left[Q \frac{dP}{dx} \right]_a.$$

Hence, find the differential coefficient of the factor that reduces to 0, multiply it by the other factor, and substitute the root in the product. The result is the same as that found by substituting the root in the second differential coefficient.

4. Let $y = (x - 3)^2(x - 2)$.

We have,

$$\frac{dy}{dx} = 2(x - 3)(x - 2) + (x - 3)^2 = (x - 3)(3x - 7) = 0;$$

$$\therefore x = 3, x = \frac{7}{3}.$$

By the principle just demonstrated,

$$\left[\frac{d^2y}{dx^2} \right]_3 = [3x - 7]_3 = +2; \left[\frac{d^2y}{dx^2} \right]_{\frac{7}{3}} = [3(x - 3)]_{\frac{7}{3}} = -2.$$

Hence, $[y]_3 = 0$, is a minimum, and $[y]_{\frac{7}{3}} = \frac{4}{27}$, is a maximum.

If y represents any function of x , and if $u = y^n$, we have,

$$\frac{du}{dx} = ny^{n-1} \frac{dy}{dx}.$$

If $x = a$ reduces $\frac{dy}{dx}$ to 0, but does not reduce y^{n-1} to 0, we have, from what precedes,

$$\left[\frac{d^2 u}{dx^2} \right]_a = \left[ny^{n-1} \frac{d^2 y}{dx^2} \right]_a.$$

If n is positive, any value that makes y a *maximum*, or *minimum*, also makes u a *maximum*, or *minimum*; if n is negative, any value that makes y a *maximum*, or *minimum*, makes u a *minimum*, or *maximum*. The converse is also true, if we except the values that make y^{n-1} equal to 0. Hence, if care be taken to reject the exceptional values, we may throw off a radical sign in seeking for a maximum or minimum.

5. Let $y = \sqrt{4a^2x^2 - 2ax^3}$.

Throwing off the radical sign, suppressing the fact $x + 2a$, and denoting the result by u , we have,

$$\frac{du}{dx} = 4ax - 3x^2 = 0; \therefore x = 0, x = \frac{4a}{3}.$$

The value, $x = 0$, makes $y = 0$, and is to be rejected.

The value $x = \frac{4}{3}a$ makes $\frac{d^2 u}{dx^2}$ *negative*, and gives

$y = \frac{8a^2}{3\sqrt{3}}$, a *maximum*.

In like manner, if we have $u = ly$, we have,

$$\frac{du}{dx} = \frac{1}{y} \cdot \frac{dy}{dx}, \text{ and if } \left[\frac{dy}{dx} \right]_a = 0, \text{ we have, as before,}$$

$$\left[\frac{d^2u}{dx^2} \right]_a = \left[\frac{1}{y} \cdot \frac{d^2y}{dx^2} \right]_a.$$

Hence, we infer, as before, that we may, with proper precaution, treat the logarithm of a function instead of the function itself, and the reverse.

$$6. \text{ Let } y = \frac{x^3 - 3x + 2}{x^3 + 3x + 2}, \text{ or } y = \frac{(x-1)(x-2)}{(x+1)(x+2)}.$$

Passing to logarithms, and denoting ly by u , we have,

$$u = l(x-1) + l(x-2) - l(x+1) - l(x+2).$$

Whence,

$$\frac{du}{dx} = \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x+1} - \frac{1}{x+2} = 0,$$

$$\text{or, } \frac{6(x^2 - 2)}{x^4 - 5x^2 + 4} = 0; \therefore x = \pm \sqrt{2}.$$

Both values of x make y negative. The first makes $\frac{d^2u}{dx^2}$ *negative*, and the second makes it *positive*. Hence we have, $y = 12\sqrt{2} - 17$, *minimum*, and $y = -12\sqrt{2} - 17$, *maximum*.

PROBLEMS IN MAXIMA AND MINIMA.

1. Divide 21 into two parts, such that the less multiplied by the square of the greater, shall be a maximum.

Solution.—Let x be the greater, and $21-x$ the less. We

have, for the equation of the problem, $y = (21 - x)x^2$. By the rule we find that $x = 14$ makes y a maximum. Hence, the parts are 14 and 7.

2. Find a cylinder whose total surface is equal to S , and whose volume is a maximum.

Solution.—Denote the radius of the base by x , the altitude by y , and the volume by V . From the geometrical relations of the parts, we have,

$$S = 2\pi x^2 + 2\pi x \times y, \text{ and } V = \pi x^2 \times y.$$

Combining, we have, for the equation of the problem,

$$V = \pi x^2 \frac{S - 2\pi x^2}{2\pi x} = \frac{1}{2} (Sx - 2\pi x^3).$$

V is a maximum when $x = \sqrt{\frac{S}{6\pi}}$; whence, $y = 2\sqrt{\frac{S}{6\pi}}$, or $y = 2x$. That is, the altitude is equal to twice the radius of the base.

3. Find the maximum rectangle that can be inscribed in an acute-angled triangle whose base is 14 feet, and altitude 10 feet.

Ans. The base is 7 feet, and the altitude 5 feet.

4. Find the maximum triangle that can be constructed on a given base, and having a given perimeter.

Solution.—Denote the base by b , a second side by x , the third side by $2p - b - x$, the perimeter by $2p$, and the area by A . From the formula for the area in terms of the three sides, we find the equation of the problem,

$$A = \sqrt{p(p-b)(p-x)(b+x-p)}.$$

For $x = p - \frac{1}{2}b$, A is a maximum. Hence, the triangle is isosceles.

5. To find the maximum isosceles triangle that can be inscribed in a circle.

Ans. An equilateral triangle.

6. To find the minimum isosceles triangle that can be circumscribed about a circle.

Ans. An equilateral triangle.

7. To find the maximum cone that can be cut from a sphere whose radius is r .

Ans. The radius of the base is $\frac{2r\sqrt{2}}{3}$, and the altitude is $\frac{4}{3}r$.

8. To find the maximum cylinder that can be cut from a cone whose altitude is h , and the radius of whose base is r .

Ans. The radius of its base is $\frac{2}{3}r$, and its altitude $\frac{1}{3}h$.

9. To find the altitude of the maximum cylinder that can be cut from a sphere whose radius is r .

Ans. Altitude = $\frac{2}{3}r\sqrt{3}$.

10. To find the maximum rectangle that can be inscribed in an ellipse whose semi-axes are a and b .

Ans. The base is $a\sqrt{2}$, and the altitude $b\sqrt{2}$.

11. To find the maximum segment of a parabola that can be cut from a right cone whose altitude is h , and the radius of whose base is r .

Ans. The axis is equal to $\frac{3}{4}\sqrt{h^2 + r^2}$.

12. To find the maximum parabola that can be inscribed in a given isosceles triangle.

Ans. The axis is equal to $\frac{3}{4}$ the altitude.

$\frac{1}{2}(h-x)$

13. To find the maximum cone whose surface is constant, and equal to S .

Ans. The radius of the base is $\frac{1}{2}\sqrt{\frac{S}{\pi}}$

14. To find the maximum cylinder that can be cut from a given oblate spheroid, whose semi-axes are a and b .

Ans. The radius of the base $= a\sqrt{\frac{2}{3}}$

and the altitude $= b\frac{2}{\sqrt{3}}$

15. From the corners of a rectangle whose sides are a and b , four squares are cut and the edges turned up to form a rectangular box. Required the side of each square when the box holds a maximum quantity.

Ans. $\frac{a+b}{6} - \frac{1}{6}\sqrt{a^2 - ab + b^2}$.

16. To find the minimum parabola that will circumscribe a given circle.

Solution.—Let x and y be the co-ordinates of the point of contact, a and b the terminal co-ordinates, $2p$ the varying parameter, and r the radius of the circle. For any circumscribed parabola, we have,

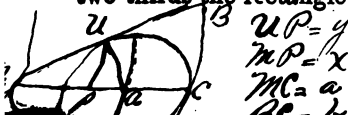
$$a = x + p + r, \text{ and } b = \sqrt{2pa} = \sqrt{2p(x + p + r)} \dots (1)$$

But from the triangle, whose sides are the radius, the subnormal, and the ordinate of the point of contact, we have,

$$y^2 = r^2 - p^2 = 2px; \therefore x = \frac{r^2 - p^2}{2p}, \quad a = \frac{(p + r)^2}{2p},$$

$$\text{and } b = p + r \dots (2)$$

It will be shown hereafter that the area of a parabola is two-thirds the rectangle having the same base and altitude.



Assuming this property, and denoting the area by A , we have, for the equation of the problem,

$$A = \frac{2}{3} \cdot 2a \cdot b = \frac{2}{3} \frac{(p+r)^2}{p} \dots\dots (3)$$

Applying the rule, and remembering that p and A are variable, we find that A is a minimum when $2p = r$.

17. Find the maximum difference between the sine and versed-sine of a varying angle.

Ans. When the arc is 45° .

18. Find the maximum value of y in the equation $y = x^{1-\frac{1}{2}}$.

Ans. $y' = x^{\frac{1}{2}}$.

19. Divide the number 36 into two factors such that the sum of their squares shall be a minimum.

Ans. Each factor is equal to 6.

20. Divide a number m into such a number of equal parts that their continued product shall be a maximum.

Solution.—Let x denote the number of parts, $\frac{m}{x}$ one part, and $\left(\frac{m}{x}\right)^x$ the continued product; hence, the equation of the problem is,

$$y = \left(\frac{m}{x}\right)^x; \text{ or, } \log y = x \log \frac{m}{x} = x \log m - x \log x.$$

$$\frac{dy}{dx} = y(\log m - 1 - \log x) = 0; \therefore \log x = \log m - 1 = \log m - \log e = \log \frac{m}{e};$$

$$\therefore x = \frac{m}{e}, \text{ and } y = \left(\frac{m}{e}\right)^{\frac{m}{e}}.$$

$$\left[\frac{d^2 y}{dx^2} \right]_{\frac{m}{e}} = \left[y \times -\frac{1}{x} \right]_{\frac{m}{e}} = -\frac{m}{e} \times \frac{e}{m}; \therefore y' = \frac{m}{e},$$

is a maximum.

21. What value of x will make the expression $\frac{\sin x}{1 + \tan x}$ a maximum?

Ans. $x = 45^\circ$.

22. Find the fraction that exceeds its square by the greatest possible quantity.

Ans. $\frac{1}{2}$.

23. The illuminating power of a ray of light falling obliquely on a plane varies inversely as the square of the distance from the source, and directly as the sine of its inclination to the plane.

How far above the centre of a horizontal circle must a light be placed that the illumination of the circumference may be a maximum?

Ans. $\pm \frac{r}{\sqrt{2}}$.

Maxima and Minima of a Function of Two Variables.

49. A function of two variables is at its *maximum* state when it is greater, and at its *minimum* state when it is less than all its consecutive states.

If two parallel sections (Art. 23) be taken through a maximum or minimum ordinate of a surface, the ordinate will be a maximum or minimum in each section, and in nearly every practical case the reverse will hold true. The exceptional cases will be those in which the maximum or minimum ordinate corresponds to a singular point of the surface. Omitting these, the problem is reduced to finding an ordinate that shall be a maximum, or a minimum

in both the parallel sections through it. This requires that $\frac{dz}{dy}$ and $\frac{dz}{dx}$ be simultaneously equal to 0. The equations thus obtained will determine all the values of x and y that correspond to maxima or minima states, and each set of such values may be tested in the manner explained in the last article, observing that for the section parallel to the plane xz , the successive differential coefficients of z , are found by supposing y constant, and for the section parallel to yz , they are found by supposing x constant.

EXAMPLES.

1. Let $z = x^2 - 3xy + y^2$.

We have,

$$\frac{dz}{dx} = 2x - 3y = 0, \text{ and } \frac{dz}{dy} = 3y - 3x = 0;$$

$$\therefore x = 0, y = 0; \text{ and } x = 1, y = 1.$$

Also,

$$\frac{d^2z}{dx^2} = 2, \frac{d^2z}{dx^2} = 2; \text{ and } \frac{d^2z}{dy^2} = 2, \frac{d^2z}{dy^2} = 2.$$

The first set of values reduce the second differential coefficients to 0, but not the third differential coefficients; hence, they are to be rejected. The second set of values make both of the second differential coefficients positive; hence, they correspond to a minimum, which is $z = -1$.

2. Let $z = x^2y^2 - x^4y^2 - x^3y^3$.

For, $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{432}$, a maximum.

3. Let $z = \sqrt{p(p-x)(p-y)(x+y-p)}$.

Dropping the radical sign, passing to logarithms, and denoting the logarithm of z^2 by u , we have,

$$u = l(p) + l(p-x) + l(p-y) + l(x+y-p).$$

Hence,

$$\frac{du}{dx} = -\frac{1}{p-x} + \frac{1}{x+y-p} = 0;$$

$$\frac{du}{dy} = -\frac{1}{p-y} + \frac{1}{x+y-p} = 0,$$

$$x = \frac{2}{3}p, y = \frac{2}{3}p, \text{ make } z \text{ a maximum.}$$

V. SINGULAR VALUES OF FUNCTIONS.

Definition and Method of Evaluation.

50. A *singular value* of a function is one that appears under an indeterminate form, for a particular value of the variable. Thus, the expression $\frac{x - \sin x}{x^3}$ reduces to $\frac{0}{0}$, for the particular value, $x = 0$, whereas its real value is $\frac{1}{6}$.

Singular values that take the above form of indetermination, may often be detected, and their real value found by means of the calculus. Let $u = \frac{z}{y}$, in which z and y are functions of x that reduce to 0 for $x = a$. Clearing of fractions and differentiating, we have,

$$ydz + ydu = dz;$$

if we make $x = a$, the second term disappears, and we have,

$$[u]_a = \left[\frac{dz}{dy} \right]_a.$$

If both dz and dy reduce to 0 for $x = a$, we have, in like manner,

$$[u]_a = \left[\frac{d^2z}{d^2y} \right]_a,$$

and so on. Hence, the following rule:

Divide the successive differentials of the numerator by the corresponding differentials of the denominator ; substitute the particular value of the variable in the resulting fraction, continuing the operation till one is reached that is not indeterminate ; the value thus found is the value required.

Thus, in the example above, we have,

$$\left[\frac{x - \sin x}{x^3} \right]_0 = \left[\frac{1 - \cos x}{3x^2} \right]_0 = \left[\frac{\sin x}{6x} \right]_0 = \left[\frac{\cos x}{6} \right]_0 = \frac{1}{6}.$$

EXAMPLES.

1. Find the value of $\left[\frac{x^3 + 2x^2 - x - 2}{x^3 - 1} \right]_1$. *Ans.* 2.
2. Find the value of $\left[\frac{x^3 - 2x}{x^4 - 2x^3 + 8x - 16} \right]_2$. *Ans.* $\frac{1}{8}$.
3. Find the value of $\left[\frac{x^3 - a^3x - ax^3 + a^3}{x^2 - a^2} \right]_a$. *Ans.* 0.
4. Find the value of $\left[\frac{1 - x^n}{1 - x} \right]_1$. *Ans.* n .
5. Find the value of $\left[\frac{e^x - e^{-x}}{l(1 + x)} \right]_0$. *Ans.* 2.
6. Find the value of $\left[\frac{1 - \cos x}{x} \right]_0$. *Ans.* 0.
7. Find the value of $\left[\frac{x^3 - 1}{x^3 + 2x^2 - x - 2} \right]_1$. *Ans.* $\frac{1}{2}$.
8. Find the value of $\left[\frac{x^4 - 1}{x^2 - 1} \right]_1$. *Ans.* 2.

9. Find the value of $\left[\frac{x^2 \tan x}{1 + \tan x} \right]_{\frac{\pi}{2}}$ *Ans.* $\frac{\pi^2}{4}$.

10. Find the value of $\left[\frac{x - \sqrt{2x^2 - a^2}}{2x - \sqrt{5x^2 - a^2}} \right]_a$. *Ans.* 2.

Singular values that appear under the forms $0 \times \infty$, $\frac{\infty}{\infty}$, and $\infty - \infty$, can be reduced to the form discussed, and then treated by the rule. The method of proceeding in each case will be illustrated by an example.

11. The expression $(1-x) \tan \frac{\pi x}{2}$, which reduces to $0 \times \infty$, when $x = 1$, can be placed under the form $\frac{1-x}{\cot \frac{\pi}{2} x}$, which, by the rule, reduces to $\frac{2}{\pi}$ for the particular value, $x = 1$.

12. The expression, $\tan \frac{\pi x}{2} \div \frac{x^2}{(x^2 - 1)}$, which reduces to $\frac{\infty}{\infty}$ for $x = 1$, can be placed under the form $\frac{x^2 - 1}{x^2 \cot \frac{\pi}{2} x}$, and this, by the rule, gives for $x = 1$, the value $-\frac{4}{\pi}$.

13. The expression, $x \tan x - \frac{\pi}{2} \sec x$, which reduces to $\infty - \infty$, for the value $x = \frac{\pi}{2}$, may be written $\frac{x \sin x - \frac{1}{2} \pi}{\cos x}$, which, by the rule, gives for $x = \frac{\pi}{2}$, the value -1 .

VI. ELEMENTS OF GEOMETRICAL MAGNITUDES.

Differentials of Lines, Surfaces, and Volumes.

51. Let AP and BQ be two consecutive ordinates of the curve, KL , whose equation is $y = f(x)$; then will PQ be the differential of the length of the curve, and $APQB$ will be the differential of the area, bounded by the curve, the axis of x , and any two ordinates; if we suppose the figure to revolve about the axis of x , the curve will generate a surface of revolution, and the area between the curve and axis will generate a volume of revolution; the surface generated by PQ , is the differential of the surface of revolution, and the volume generated by $APBQ$, is the differential of the volume of revolution. When the equation of KL is given, the values of these differentials may always be found in terms of x and dx , or of y and dy .

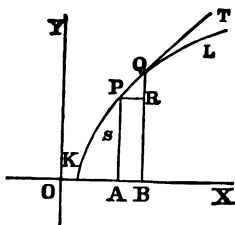


Fig. 11.

1°. Denote PQ by dL ; we shall have, as in Art. 5,

$$dL = \sqrt{dx^2 + dy^2} \dots (1)$$

To apply this formula, we differentiate the equation of the curve, combine the resulting equation with that of the curve, so as to find the differential of one variable in terms of the other variable and its differential, and substitute this in the formula. Thus, let it be required to find the differential of the arc of a parabola. Assuming the equation of the parabola, $y^2 = 2px$, we find, by differentiation and combination,

$$dy = dx\sqrt{\frac{p}{2x}}, \text{ and } dx = \frac{y}{p}dy, \text{ or } dy^2 = dx^2\frac{p}{2x},$$

$$\text{and } dx^2 = \frac{y^2}{p^2}dy^2.$$

Substituting, and reducing, we have,

$$dL = dx\sqrt{\frac{2x+p}{2x}}, \text{ or } dL = \frac{dy}{p}\sqrt{p^2 + y^2}.$$

2°. Denote $APQB$ by dA ; this is made up of two parts, the rectangle, $AR = ydx$, and the triangle, $RPQ = \frac{1}{2}dydx$; but the latter is an infinitesimal of the second order, it may be therefore neglected in comparison with the former; hence, we have, •

$$dA = ydx \dots \dots (2)$$

This formula may be applied in a manner similar to that just explained. Thus, if it be required to find the differential of the area of a parabola, we have,

$$dA = \frac{y^2}{p}dy; \text{ or, } dA = (\sqrt{2px})dx.$$

3°. Denote the surface generated by PQ by dS ; this surface is that of a frustum of a cone, in which the radius of the upper base is y , the radius of the lower base, $y + dy$, and the slant height, $\sqrt{dx^2 + dy^2}$. Hence,

$$dS = \frac{1}{2}[2\pi y + 2\pi(y + dy)] \times \sqrt{dx^2 + dy^2}.$$

Neglecting dy in comparison with y , and reducing, we have,

$$dS = 2\pi y\sqrt{dx^2 + dy^2} \dots \dots (3)$$

To find the differential of a paraboloid, we proceed as before. Substituting the values already found, and reducing, we have,

$$dS = 2\pi dx \sqrt{2px + p^2}; \text{ or, } dS = \frac{2\pi y dy}{p} \sqrt{p^2 + y^2}.$$

4°. Denote the volume generated by $APQB$, by dV ; this volume is that of a frustum of a cone, in which the radius of the upper base is y , of the lower base, $y + dy$, and the altitude dx . Hence,

$$dV = \frac{[\pi y^2 + \pi(y + dy)^2 + \pi y(y + dy)]dx}{3};$$

neglecting dy in comparison with y , we have,

$$dV = \pi y^2 dx \dots \dots (4)$$

Applying the formula to the paraboloid of revolution, we have, as before,

$$dV = \pi \frac{y^2}{p} dy, \text{ or, } dV = 2\pi p x dx.$$

VII. APPLICATION TO POLAR CO-ORDINATES.

General Notions, and Definitions.

52. In a polar system, the radius vector is usually taken as the *function*, the angle being the *independent variable*. Let P be any point of the curve, PL , in the plane of the axes, OX , OY ; let O be the *pole*, and OX the *initial line*, of a system of polar co-ordinates. Then will OP ,

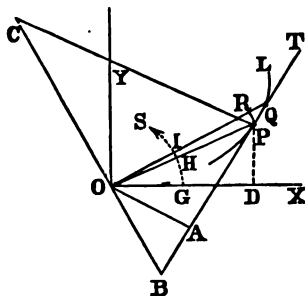


Fig. 12.

denoted by r , and XOP , denoted by φ , be the polar coordinates of P , and the equation of PL may be written under the form,

$$r = f(\varphi).$$

It will be convenient to express the values of φ in terms of π as a unit; in this case, it is laid off on the *directing circle*, GS , in the direction of the arrow when positive, and in a contrary direction when negative. Let GH be the measure of φ for the point P , and let HI be equal to the constant infinitesimal, $d\varphi$; draw OIQ , and with O as a centre describe the arc, PR ; then will OP and OQ be consecutive radius vectors, P and Q will be consecutive points, RQ will be the differential of r , PQ will be the differential of the arc, POQ will be the differential of the area swept over by the radius vector, and PQ prolonged, will be tangent to the curve at P .

The line, BC , perpendicular to the radius vector, OP , is the *movable axis*, and the perpendicular distance, OA , is the *polar distance of the tangent*. Prolong the tangent to meet the movable axis at B ; at P draw a normal, and produce it to meet the movable axis at C ; then is OB the *subtangent*, PB the *tangent*, OC the *subnormal*, and PC the *normal* at the point P .

Useful Formulas.

53. 1°. *Differential of the arc.* Denote the differential of the arc, PQ , by dL ; RP being infinitesimal may be regarded as a straight line perpendicular to OQ ; it is equal to $rd\varphi$; hence,

$$dL = \sqrt{dr^2 + r^2 d\varphi^2} \dots \dots (1)$$

2°. *Differential of the area.* Denote the differential of the area swept over by the radius vector by dA ; this is made up of the sector, OPR , and the triangle, RPQ ; but, RPQ is an infinitesimal of the second order, and the sector is an infinitesimal of the first order; neglecting the former in comparison with the latter, and remembering that the area of a sector is equal to half the product of its arc and radius, we have,

$$dA = \frac{1}{2}r^2 d\phi \dots\dots (2)$$

3°. *Angle between the radius vector and tangent.* Denote the required angle by V ; the angle, RQP , differs from the required angle, OPB , by an infinitesimal, and may, therefore, be taken for it; but $\tan RQP$, equals RP , divided by RQ ; hence, we have,

$$\tan V = \frac{r d\phi}{dr} \dots\dots (3)$$

4°. *Polar distance of the tangent.* Denote the distance OA by p ; the triangles, QPR and QOA , are similar; hence,

$$QP : RP :: QO : OA.$$

But, PO differs from QO by an infinitesimal, and may, therefore, be taken for it; making this change, and substituting for the quantities their values, we have,

$$\sqrt{dr^2 + r^2 d\phi^2} : r d\phi :: r : p;$$

hence,

$$p = \frac{r^2 d\phi}{\sqrt{dr^2 + r^2 d\phi^2}} \dots\dots (4)$$

5°. *Formula for subtangent.* Denote the subtangent by $S.T$; in the triangle, POB , the perpendicular, OB , is equal to the base, OP , multiplied by the tangent of the angle at the base; hence,

$$S.T = \frac{r^2 d\phi}{dr} \dots \dots (5)$$

6°. *Formula for subnormal.* Denote the subnormal by $S.N$; the triangles, OPB and OCP , are similar; hence, $\tan OCP = \tan OPB$; but, OC , equals OP , divided by $\tan OCP$; hence,

$$S.N = \frac{dr}{d\phi} \dots \dots (6)$$

7°. *Formula for the radius of curvature.* Denote the radius of curvature by ρ ; let P and Q be two consecutive points of the curve, KL , and let PC and QC be normals at these points, meeting at C ; then will C be the centre of the osculatory circle; draw OM perpendicular to PC , it will be parallel to the tangent at P , and, consequently, the intercept, PM , will be equal to the polar distance of the tangent denoted by p ; draw OC , OP , and OQ . From the figure, we have,

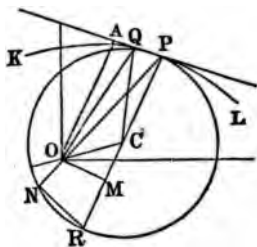


Fig. 12.

$$\overline{OC}^2 = r^2 + \rho^2 - 2p\rho.$$

If we pass from P to Q , r will become $r + dr$, p will become $p + dp$, and OC and ρ will remain unchanged. If, therefore, we differentiate the preceding equation under the supposition that r and p are variable, the resulting

equation will express a relation between ρ , r , and p . Differentiating, we have,

$$2rdr - 2\rho dp = 0; \therefore \rho = \frac{rdr}{dp} \dots\dots (7)$$

8°. *Formula for the chord of curvature.* The chord of curvature is the chord of the osculatory circle that passes through the pole and the point of osculation. Denote it by C ; prolong PC and PO till they meet the circle at R and N , and draw RN . The right angled triangles, PNR and PMO , are similar, hence,

$$PN : PR :: PM : PO,$$

or,

$$C : 2\rho :: p : r;$$

hence,

$$C = \frac{2p\rho}{r}, \text{ or } C = 2p \frac{dr}{dp} \dots\dots (8)$$

Spirals.

54. If a straight line revolve uniformly about one of its points as a centre, and if, at the same time, a second point travel along the line, in accordance with any law, the latter point will describe a *spiral*. The part described in one revolution of the straight line is a *spire*. The fixed point is the *pole*. If we denote the distance from the pole to the generating point by r , and the corresponding angle, counted from the initial line, by ϕ , we have, for the general equation of spirals,

$$r = f(\phi) \dots\dots (1)$$

When this relation is algebraic, the spiral is said to be *algebraic*; when this relation is transcendental, the spiral is said to be *transcendental*.

Among algebraic spirals, the most important are the *spiral of Archimedes*, the *parabolic spiral*, and the *hyperbolic spiral*, corresponding, respectively, to the *right line*, the *parabola*, and the *hyperbola*.

Spiral of Archimedes.

55. The equation of this curve is,

$$r = a\varphi \dots\dots (2)$$

We see that the generating point is at the pole when the variable angle is 0, and that the radius vector increases uniformly with the variable angle, being always equal to a times the arc of the directing circle that has been swept over. Differentiating equation (2), we have,

$$dr = a d\varphi.$$

Substituting, in formulas (1) to (8), Art. 53, we find,

$$dL = a d\varphi \sqrt{1 + \varphi^2}; \quad S.T = a\varphi^2;$$

$$dA = \frac{1}{2} a^2 \varphi^2 d\varphi; \quad S.N = a;$$

$$\text{Tan } V = \frac{r}{a}; \quad \rho = \frac{a(1 + \varphi^2)^{\frac{3}{2}}}{2 + \varphi^2}.$$

$$p = \frac{a\varphi^2}{\sqrt{1 + \varphi^2}}; \quad C = \frac{2a\varphi(1 + \varphi^2)}{2 + \varphi^2}.$$

If we take a straight line, whose equation is $y = ax$, and lay off the abscissa of any point on the directing circle, and the ordinate of that point on the corresponding radius vector, the point thus determined is a point of the spiral of Archimedes.

In like manner, we may discuss and construct the parabolic spiral, whose equation is $r^2 = 2p\phi$, and the hyperbolic spiral, whose equation is $r\phi = m$. The former corresponds to the ordinary parabola, and the latter, to the ordinary hyperbola referred to its asymptotes.

VIII. TRANSCENDENTAL CURVES.

Definition.

56. A transcendental curve, is a curve whose equation can only be expressed by the aid of transcendental quantities. The cycloid, and the logarithmic curve are examples of this class of lines.

The Cycloid.

57. The cycloid is a curve that may be generated by a point in the circumference of a circle, when that circle rolls along a straight line. The point is called the *generatrix*, the circle is called the *generating circle*, and the straight line is the *base* of the cycloid. The curve has an infinite number of branches, each corresponding to one revolution of the generating circle.

To find the equation of one branch, APM ; let A be the origin, C the centre of the generating circle, and P the place of the generatrix after the circle has rolled through the angle KCP , denoted by ϕ ; denote the co-ordinates, AL and LP , by x and y , and let the radius of the generating circle be represented by r .

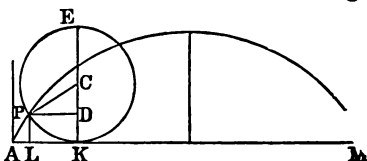


Fig. 14.

From the figure, we have,

$$AL = AK - LK.$$

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But, AK is equal to the arc KP , and KP is the arc whose versed sine is y , to the radius r , or r times the arc whose versed sine is $\frac{y}{r}$, to the radius 1; LK is equal to PD , which, from a property of the circle, is equal to $\pm \sqrt{(2r - y)y}$, the upper sign corresponding to the case in which P is to the left of KE , and the lower sign to the case in which P is to the right of KE ; and AL is equal to x . Substituting these values, and reducing, we have,

$$x = r \text{versin}^{-1} \frac{y}{r} \mp \sqrt{2ry - y^2};$$

which is the equation sought. From it we see that y can never be less than 0, nor greater than $2r$; we see, also, that y is equal to 0, for $x = 0$, and for $x = 2\pi r$, and that for every value of y , there are two values of x , one exceeding πr as much as the other falls short of it.

Differentiating and reducing, we have,

$$dx = \pm \frac{y dy}{\sqrt{2ry - y^2}},$$

which is the differential equation of the cycloid. From it we deduce the equation,

$$\frac{dy}{dx} = \pm \sqrt{\frac{2r}{y} - 1}, \text{ whence, } \frac{d^2y}{dx^2} = -\frac{r}{y^2}.$$

From the first, we see that the tangent to the curve is vertical for $x = 0$, and $x = 2\pi r$, and that it is horizontal for $x = \pi r$; from the second, we see that the curve is always concave downward.

Substituting, in formulas (6) and (7), Art. 32, we have,

$$S.T = \pm \frac{y''^2}{\sqrt{2ry'' - y''^2}}, \text{ and } S.N = \mp \sqrt{2ry'' - y''^2}.$$

The Logarithmic Curve.

58. The logarithmic curve, is a curve in which any ordinate is equal to the logarithm of the corresponding abscissa. Its equation is, therefore,

$$y = \log x, \text{ or } y = M \cdot lx;$$

in which M is the modulus of the system.

If $M > 0$, y will be negative when $x < 1$, positive when $x > 1$, 0 when $x = 1$; and infinite when $x = 0$.

If $M < 0$, y will be positive when $x < 1$, negative when $x > 1$, 0 when $x = 1$, and infinite when $x = 0$.

In no case can a negative value of x correspond to a point of the curve.

Differentiating, we find,

$$\frac{dy}{dx} = \frac{M}{x}, \text{ and } \frac{d^2y}{dx^2} = -\frac{M}{x^2}.$$

The first expression has the same sign as M , and varies inversely as x . Hence, when $M > 0$, the slope is positive, when $M < 0$, the slope is negative, and in both cases the curve continually approaches parallelism with the axis of x .

The second expression has a sign contrary to that of M , and varies inversely as the square of x . Hence, when $M > 0$, the curve is concave downward, as KDC , and when $M < 0$, it is concave upward, as KDC' .

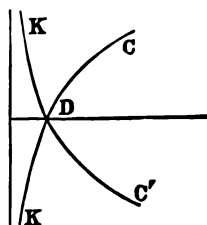


Fig. 15.

PART III.

INTEGRAL CALCULUS.

Object of the Integral Calculus.

X 59. The object of the integral calculus is to pass from a given differential to a function from which it may have been derived. This function is called the *integral of the differential*, and the operation of finding it is called *integration*. The operation of integration is indicated by this sign, $\int$, called the *integral sign*. Integration and differentiation are inverse operations, and their signs, when placed before a quantity, neutralize each other; thus, $\int dx = x$.

A constant quantity connected with a function by the sign of addition, or subtraction, disappears by differentiation; hence, we must add a constant to the integral obtained; and inasmuch as this constant may have any value, it is said to be *arbitrary*. The integral, before the addition of the constant, is called *incomplete*, after its addition it is said to be *complete*. By means of the arbitrary constant, as we shall see hereafter, the integral may be made to satisfy any one reasonable condition.

Nature of an Integral.

60. The differential of a function is the difference between two consecutive states of that function; hence,

any state of the function whatever, is equal to some previous, or *initial* state, *plus* the algebraic sum of all the intermediate values of the differential. But this state is the integral, by definition; hence, if the arbitrary constant represents the initial state, the incomplete integral represents the algebraic sum of all the differentials *from* the initial state *up to* any state whatever, and the complete integral represents the initial state, *plus* the algebraic sum of all the differentials from that state up to any state whatever.

Before any value has been assigned to the constant, the integral is said to be *indefinite*; when the value of the constant has been determined, so as to satisfy a particular hypothesis, the integral is said to be *particular*; and when a definite value has been given to the variable, this integral is said to be *definite*.

The values of the variable that correspond to the *initial* and *terminal* states of a definite integral are called *limits*, the former being the *inferior*, and the latter the *superior limit*; the integral is said to lie between these limits. The value of the definite integral may be found from the indefinite integral by substituting for the variable the *inferior* and *superior limits* separately, and then taking the first result from the second. The first result is the integral from any state up to the first limit, and the second is the integral from the same state up to the last limit; hence, the difference is the integral between the limits. The symbol for integrating between the limits is $\int_a^b$, in which a is the inferior, and b the superior limit.

This article will be better understood when we come to its practical application in Articles 78 and following.

Methods of Integration; Simplifications.

61. The methods of integration are not founded on processes of direct reasoning, but are mostly dependent on formulas. The more elementary formulas are deduced by reversing corresponding formulas in the differential calculus; the more complex ones are deduced from these by various transformations and devices, whereby the expressions to be integrated are brought under some known integrable form.

It has been shown (Art. 9), that a constant factor remains unchanged by differentiation; hence, a constant factor may be placed without the sign of integration. It was also shown in the same article, that the differential of the sum of any number of functions is equal to the sum of their differentials; hence, the integral of the sum of any number of differentials is equal to the sum of their integrals. These principles are of continued use in integration.

Fundamental Formulas.

62. If we differentiate the expression $\frac{ax^{n+1}}{n+1}$, we have,

$$d\left(\frac{ax^{n+1}}{n+1}\right) = ax^n dx;$$

reversing the formula, and applying the integral sign to both members,

$$\int ax^n dx = \int d\left(\frac{ax^{n+1}}{n+1}\right).$$

Remembering that the signs of integration and differentia-

tion neutralize each other (Art. 59), and adding a constant to complete the integral, we have,

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \dots\dots [1]$$

Hence, to integrate a monomial differential,—*drop the differential of the variable, add 1 to the exponent of the variable, divide the result by the new exponent, and add a constant.*

This rule fails when $n = -1$; for, if we apply it, we get a result equal to ∞ ; in this case the integration may be effected as follows:

From Art. 15, we have,

$$d(ax) = a \frac{dx}{x} = ax^{-1} dx;$$

reversing and proceeding as before, we have,

$$\int ax^{-1} dx = a \int \frac{dx}{x} = ax + C \dots\dots [2]$$

From Art. 16, we have,

$$d\left(\frac{a^x}{\ln a}\right) = a^x dx;$$

reversing and proceeding as before, we have,

$$\int a^x dx = \frac{a^x}{\ln a} + C \dots\dots\dots [3]$$

In like manner, reversing the formulas in Article 17, lettered from a to h , we have,

$$\int \cos x dx = \sin x + C \dots\dots\dots [4]$$

$$\int -\sin x \, dx = \cos x + C \dots\dots\dots [5]$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C \dots\dots\dots [6]$$

$$\int -\frac{dx}{\sin^2 x} = \cot x + C \dots\dots\dots [7]$$

$$\int \sin x \, dx = -\cos x + C \dots\dots\dots [8]$$

$$\int -\cos x \, dx = \sin x + C \dots\dots\dots [9]$$

$$\int \tan x \sec x \, dx = \sec x + C \dots\dots\dots [10]$$

$$\int -\cot x \operatorname{cosec} x \, dx = \operatorname{cosec} x + C \dots\dots\dots [11]$$

In like manner, from the formulas in Art. 18, lettered from a' to h' , we deduce the following:

$$\int \frac{dy}{\sqrt{1-y^2}} = \sin^{-1} y + C \dots\dots\dots [12]$$

$$\int -\frac{dy}{\sqrt{1-y^2}} = \cos^{-1} y + C \dots\dots\dots [13]$$

$$\int \frac{dy}{1+y^2} = \tan^{-1} y + C \dots\dots\dots [14]$$

$$\int -\frac{dy}{1+y^2} = \cot^{-1} y + C \dots\dots\dots [15]$$

$$\int \frac{dy}{\sqrt{2y - y^2}} = \text{versin}^{-1}y + C \dots\dots [16]$$

$$\int -\frac{dy}{\sqrt{2y - y^2}} = \text{coversin}^{-1}y + C \dots [17]$$

$$\int \frac{dy}{y\sqrt{y^2 - 1}} = \sec^{-1}y + C \dots\dots\dots [18]$$

$$\int -\frac{dy}{y\sqrt{y^2 - 1}} = \text{cosec}^{-1}y + C \dots\dots [19]$$

If in formulas (12), (13), (14), (15), (18), (19), we make

$y = \frac{bx}{a}$, whence $dy = \frac{b dx}{a}$, and reduce, we have,

$$\int \frac{dx}{\sqrt{a^2 - b^2 x^2}} = \frac{1}{b} \sin^{-1} \frac{bx}{a} + C \dots\dots [20]$$

$$\int -\frac{dx}{\sqrt{a^2 - b^2 x^2}} = \frac{1}{b} \cos^{-1} \frac{bx}{a} + C \dots [21]$$

$$\int \frac{dx}{a^2 + b^2 x^2} = \frac{1}{ab} \tan^{-1} \frac{bx}{a} + C \dots\dots [22]$$

$$\int -\frac{dx}{a^2 + b^2 x^2} = \frac{1}{ab} \cot^{-1} \frac{bx}{a} + C \dots [23]$$

$$\int \frac{dx}{x\sqrt{b^2 x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{bx}{a} + C \dots\dots [24]$$

$$\int -\frac{dx}{x\sqrt{b^2x^2 - a^2}} = \frac{1}{a} \operatorname{cosec}^{-1} \frac{bx}{a} + C. \quad \dots [25]$$

From (16) and (17), by substituting $\frac{2b^2}{a^2}x$ for y , and reducing, we have,

$$\int \frac{dx}{\sqrt{a^2x - b^2x^2}} = \frac{1}{b} \operatorname{versin}^{-1} \frac{2b^2}{a^2}x + C \dots \dots [26]$$

$$\int -\frac{dx}{\sqrt{a^2x - b^2x^2}} = \frac{1}{b} \operatorname{coversin}^{-1} \frac{2b^2}{a^2}x + C \dots [27]$$

Formulas (1) to (19) are the elementary forms, to one of which we endeavor to reduce every case of integration; the processes of the integral calculus are little else than a succession of transformations and devices, by which this reduction is affected.

In the following examples, and, as a general thing, throughout the book, the incomplete integral is given, it being understood that an arbitrary constant is to be added in every case.

EXAMPLES.

Formulas (1) to (3).

$$1. \quad dy = x^3 dx. \qquad \qquad \qquad \text{Ans. } y = \frac{ax^4}{4}.$$

$$2. \quad dy = \sqrt{x} dx. \qquad \qquad \qquad \text{Ans. } y = \frac{2}{3} bx^{\frac{3}{2}}.$$

$$3. \quad dy = 3x^{-4} dx. \qquad \qquad \qquad \text{Ans. } y = -x^{-3}.$$

$$4. \, dy = 2x^{\frac{5}{4}} dx. \qquad \text{Ans. } y = \frac{7}{6} x^{\frac{9}{4}}.$$

$$5. \, dy = -\frac{2}{3} x^{-\frac{4}{3}} dx. \qquad \text{Ans. } y = \frac{10}{3} x^{-\frac{1}{3}}.$$

$$6. \, dy = \left(\frac{5}{2} ax^{\frac{3}{2}} - \frac{3}{2} bx^{\frac{1}{2}} \right) dx. \qquad \text{Ans. } y = ax^{\frac{5}{2}} - bx^{\frac{3}{2}}.$$

$$7. \, dy = \left(\frac{12}{x^2} - \frac{3}{x^4} \right) dx. \qquad \text{Ans. } y = -\frac{12}{x} + \frac{1}{x^3}.$$

$$8. \, dy = \frac{1}{2} x^{-\frac{1}{2}} dx. \qquad \text{Ans. } y = x^{\frac{1}{2}}.$$

$$9. \, dy = 3x^{-1} dx. \qquad \text{Ans. } y = 3 \ln x.$$

$$10. \, dy = (2x^{\frac{2}{3}} + x^{-1}) dx. \qquad \text{Ans. } y = \frac{6}{5} x^{\frac{5}{3}} + \ln x.$$

$$11. \, dy = e^{\sin x} \cos x dx. \qquad \text{Ans. } y = e^{\sin x}.$$

$$12. \, dy = -e^{\cos x} \sin x dx. \qquad \text{Ans. } y = e^{\cos x}.$$

Formulas (4) to (11).

$$13. \, dy = 2 \cos 2x dx. \qquad \text{Ans. } y = \sin 2x.$$

$$14. \, dy = -\frac{1}{2} \sin(x^2) \times x dx. \qquad \text{Ans. } y = \frac{1}{4} \cos(x^2).$$

$$15. \, dy = \frac{dx}{\cos^2(\frac{1}{2}x)}. \qquad \text{Ans. } y = 2 \tan\left(\frac{1}{2}x\right).$$

$$16. \, dy = -\frac{x dx}{\sin^2(3x^2)}. \qquad \text{Ans. } y = \frac{1}{6} \cot(3x^2).$$

$$17. \, dy = \sin(ax)dx. \qquad \text{Ans. } y = \frac{1}{a} \text{versin}(ax).$$

$$18. \, dy = -\cos\left(\frac{1}{2}x^2\right) \times xdx. \text{ Ans. } y = \text{coversin}\left(\frac{1}{2}x^2\right).$$

Formulas (12) to (19).

$$19. \, dy = \frac{2dx}{\sqrt{1-4x^2}}. \qquad \text{Ans. } y = \sin^{-1}(2x).$$

$$20. \, dy = \frac{2xdx}{\sqrt{1-x^4}}. \qquad \text{Ans. } y = \sin^{-1}(x^2).$$

$$21. \, dy = \frac{x^{\frac{1}{2}}dx}{\sqrt{2-4x^3}} = \frac{1}{3} \frac{d(x\sqrt{2x})}{\sqrt{1-2x^3}}.$$

$$\text{Ans. } y = \frac{1}{3} \sin^{-1}(x\sqrt{2x}).$$

$$22. \, dy = -\frac{dx}{\sqrt{x-4x^3}} = -\frac{d(2x^{\frac{1}{2}})}{\sqrt{1-4x}}.$$

$$\text{Ans. } y = \cos^{-1}(2\sqrt{x}).$$

$$23. \, dy = \frac{xdx}{1+x^4} = \frac{1}{2} \frac{d(x^2)}{1+x^4}. \qquad \text{Ans. } y = \frac{1}{2} \tan^{-1}x^2.$$

$$24. \, dy = -\frac{2dx}{1+x^2}. \qquad \text{Ans. } y = 2\cot^{-1}x.$$

$$25. \, dy = \frac{dx}{\sqrt{6x-9x^3}}. \qquad \text{Ans. } y = \frac{1}{3} \text{versin}^{-1}3x.$$

Formulas (20) to (25).

$$26. \, dy = \frac{dx}{\sqrt{2-9x^2}}; \quad (a = \sqrt{2}, \, b = 3).$$

$$Ans. \, y = \frac{1}{3} \sin^{-1} \frac{3x}{\sqrt{2}}.$$

$$27. \, dy = \frac{-\frac{2}{3}x dx}{\sqrt{2-5x^4}}; \quad (a = \sqrt{2}, \, b = \sqrt{5}, \, u = x^2).$$

$$Ans. \, y = \frac{1}{3\sqrt{5}} \cos^{-1} \frac{x^2\sqrt{5}}{\sqrt{2}}.$$

$$28. \, dy = \frac{3dx}{4+9x^2}; \quad (a = 2, \, b = 3).$$

$$Ans. \, y = \frac{1}{2} \tan^{-1} \frac{3x}{2}.$$

$$29. \, dy = -\frac{2dx}{4+x^2}; \quad (a = 2, \, b = 1).$$

$$Ans. \, y = \cot^{-1} \frac{x}{2}.$$

$$30. \, dy = \frac{2dx}{x\sqrt{3x^2-5}}; \quad (a = \sqrt{5}, \, b = \sqrt{3}).$$

$$Ans. \, y = \frac{2}{\sqrt{5}} \sec^{-1} \frac{x\sqrt{3}}{\sqrt{5}}.$$

$$31. \, dy = -\frac{3dx}{\sqrt{5x^4-2x^2}}; \quad (a = \sqrt{2}, \, b = \sqrt{5}).$$

$$Ans. \, y = \frac{3}{\sqrt{2}} \operatorname{cosec}^{-1} \frac{x\sqrt{5}}{\sqrt{2}}.$$

$$32. \, dy = \frac{dx}{1 + 5x^2}, \quad \text{Ans. } y = \frac{1}{\sqrt{5}} \tan^{-1} x \sqrt{5}.$$

$$33. \, dy = \frac{-2x^{-1}dx}{\sqrt{14x^2 - 3}}, \quad \text{Ans. } y = \frac{2}{\sqrt{3}} \operatorname{cosec}^{-1} \frac{x\sqrt{14}}{\sqrt{3}}.$$

Formulas (26) and (27).

$$34. \, dy = \frac{2xdx}{\sqrt{4x^2 - 9x^4}}; \quad (a = 2, \, b = 3).$$

$$\text{Ans. } y = \frac{1}{3} \operatorname{versin}^{-1} \frac{9x^2}{2}.$$

$$35. \, dy = \frac{-3axdx}{\sqrt{x^2 - 2x^4}}; \quad (a = 1, \, b = \sqrt{2}).$$

$$\text{Ans. } y = \frac{3a}{2\sqrt{2}} \operatorname{coversin}^{-1} 4x^2.$$

EXAMPLES IN SUCCESSIVE INTEGRATION.

$$36. \, d^3y = ax \, dx^2.$$

Dividing by dx^2 ,

$$\frac{d^3y}{dx^2} = ax \, dx, \text{ or } d\left(\frac{d^2y}{dx^2}\right) = ax \, dx.$$

Integrating, we have,

$$\frac{d^2y}{dx^2} = \frac{ax^2}{2} + C.$$

Multiplying by dx , and integrating again, we find,

$$\frac{dy}{dx} = \frac{ax^3}{6} + Cx + C'.$$

Multiplying again by dx , and integrating,

$$y = \frac{ax^4}{24} + \frac{Cx^3}{2} + C'x + C''.$$

$$37. d^4y = -x^{-4}dx^4.$$

As before, we have,

$$\frac{d^3y}{dx^3} = \frac{x^{-3}}{3} + C;$$

$$\frac{d^2y}{dx^2} = -\frac{x^{-2}}{6} + Cx + C';$$

$$\frac{dy}{dx} = \frac{x^{-1}}{6} + \frac{Cx^2}{2} + C'x + C'';$$

and finally,

$$y = \frac{1}{6}lx + \frac{Cx^3}{6} + \frac{C'x^2}{2} + C''x + C'''.$$

$$38. d^3y = m dx^3.$$

$$Ans. y = \frac{mx^3}{6} + \frac{Cx^2}{2} + C'x + C''.$$

$$39. d^3y = x^{\frac{1}{2}} dx^3.$$

$$Ans. y = \frac{8}{105}x^{\frac{7}{2}} + \frac{Cx^2}{2} + C'x + C''.$$

$$40. d^3y = sx^2 dx^3.$$

$$Ans. y = \frac{sx^4}{12} + Cx + C'.$$

Integration by Parts.

63. If u and v are functions of x , we have, from Art. 10,

$$d(uv) = u dv + v du.$$

Integrating and transposing, we have,

$$\int u dv = uv - \int v du \dots\dots [28]$$

This is called the *formula for integration by parts*; it enables us to integrate an expression of the form $u dv$ whenever we can integrate an expression of the form $v du$. It is much used in reducing integrals to known forms.

EXAMPLES.

1. $dy = x^2 dx$;

Let $lx = u$, and $x dx = dv$;

$$\therefore du = \frac{dx}{x}, \text{ and } v = \frac{x^2}{2}.$$

Substituting in the formula, we have,

$$y = \frac{x^2 lx}{2} - \int \frac{x dx}{2} = \frac{x^2 lx}{2} - \frac{x^2}{4}.$$

2. $dy = \frac{dx \sqrt{1-x^2}}{x^2}$

Let $\sqrt{1-x^2} = u$, and $\frac{dx}{x^2} = dv$;

$$\therefore du = -\frac{xdx}{\sqrt{1-x^2}}, \text{ and } v = -\frac{1}{x};$$

hence, we have, from the formula,

$$y = -\frac{\sqrt{1-x^2}}{x} - \int \frac{dx}{\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x.$$

Additional Formulas.

64. 1°. Formula (1) may be extended to cover the case of a binomial differential of the form,

$$dy = (a + bx^n)^p x^{n-1} dx,$$

in which the exponent of the variable without the parenthesis is 1 less than the exponent of the variable within.

$$\text{Let } a + bx^n = z; \therefore bnx^{n-1}dx = dz, \text{ or } x^{n-1}dx = \frac{dz}{bn}.$$

Substituting, and integrating, we have,

$$y = \int (a + bx^n)^p x^{n-1} dx = \int \frac{z^p dz}{bn} = \frac{z^{p+1}}{bn(p+1)}.$$

Replacing z by its equal, $(a + bx^n)$, we have,

$$\int (a + bx^n)^p x^{n-1} dx = \frac{(a + bx^n)^{p+1}}{bn(p+1)} + C \dots \dots [29]$$

Hence, to integrate a binomial differential of the proposed form, *add 1 to the exponent of the parenthesis, divide the result by the new exponent into the product of the coefficient and the exponent of the variable within the parenthesis.*

It is to be observed that a constant factor may be set aside during the process of integration, and then introduced, as explained in Article 61.

EXAMPLES.

$$1. \, dy = (1 + x^2)^{\frac{1}{2}} x dx. \quad \text{Ans. } y = \frac{(1 + x^2)^{\frac{3}{2}}}{3}$$

$$2. \, dy = \frac{x dx}{\sqrt{a^2 + x^2}}. \quad \text{Ans. } y = \sqrt{a^2 + x^2}$$

$$3. \, dy = 5x^3 dx \sqrt{7 + 3x^4}. \quad \text{Ans. } y = \frac{5}{18} (7 + 3x^4)^{\frac{3}{2}}$$

$$4. \, dy = (2 + 3x^2)^3 x dx. \quad \text{Ans. } y = \frac{1}{24} (2 + 3x^2)^4$$

$$5. \, dy = \frac{2x dx}{(x^2 + 1)^2}. \quad \text{Ans. } y = -\frac{1}{(x^2 + 1)}$$

2°. The preceding formula fails when $p = -1$; in that case, we have,

$$\int (a + bx^n)^{-1} x^{n-1} dx = \int \frac{z^{-1} dz}{bn} = \frac{1}{bn} \int \frac{dz}{z} = \frac{1}{bn} l z + C.$$

Replacing z by its value,

$$\int (a + bx^n)^{-1} x^{n-1} dx = \frac{1}{bn} l(a + bx^n) + C \dots \dots [30]$$

EXAMPLES.

$$1. \, dy = 3(3 + 4x^2)^{-1} x^2 dx. \quad \text{Ans. } y = \frac{1}{4} l(3 + 4x^2).$$

$$2. \, dy = \frac{dx}{x + a}. \quad \text{Ans. } y = l(x + a).$$

$$3. \, dy = \frac{2dx}{x + 4}. \quad \text{Ans. } y = 2 l(x + 4)$$

When the differential expression is a fraction in which the numerator is equal to the differential of the denominator multiplied by a constant, its integral is equal to that constant into the Napierian logarithm of the denominator.

$$4. \quad dy = \frac{x^2 dx}{a^3 - x^3}. \quad \text{Ans. } y = -\frac{1}{3} \log(a^3 - x^3).$$

$$5. \quad dy = \frac{3x^2 + 2x + 1}{x^3 + x^2 + x + 1} dx. \quad \text{Ans. } y = \log(x^3 + x^2 + x + 1).$$

3°. Let it be required to integrate the expression,

$$dy = \frac{dx}{\sqrt{x^2 \pm a^2}};$$

☞

assume,

$$x^2 \pm a^2 = z^2; \text{ whence, } xdx = zdz;$$

adding zdx to both members and factoring, we have,

$$(x + z)dx = z(dx + dz);$$

hence,

$$\frac{dx}{z} = \frac{dx}{\sqrt{x^2 \pm a^2}} = \frac{dx + dz}{x + z} = \frac{d(x + z)}{x + z};$$

applying the principle just deduced, we have,

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + z) + C.$$

Replacing z by its value, we have, finally,

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + C \dots \dots [31]$$

EXAMPLES.

$$1. \quad dy = \frac{dx}{\sqrt{4x^2 - 7}} = \frac{1}{2} \frac{dx}{\sqrt{x^2 - \frac{7}{4}}}$$

$$Ans. \quad y = \frac{1}{2} l(x + \sqrt{x^2 - \frac{7}{4}}).$$

$$2. \quad dy = \frac{3dx}{\sqrt{x^2 - 5}}.$$

$$Ans. \quad y = 3l(x + \sqrt{x^2 - 5}).$$

4°. Let it be required to integrate the expression,

$$dy = \frac{dx}{\sqrt{2ax + x^2}};$$

since, $dx = d(a + x)$, and $2ax + x^2 = (a + x)^2 - a^2$, we have,

$$\int \frac{dx}{\sqrt{2ax + x^2}} = \int \frac{d(a + x)}{\sqrt{(a + x)^2 - a^2}},$$

which can be integrated by Formula (31); hence,

$$\int \frac{dx}{\sqrt{2ax + x^2}} = l \left\{ a + x + \sqrt{2ax + x^2} \right\} + C \dots [32]$$

EXAMPLES.

$$1. \quad dy = \frac{2dx}{\sqrt{3x + 4x^2}} = \frac{dx}{\sqrt{\frac{3}{4}x + x^2}}.$$

$$Ans. \quad y = l \left(x + \frac{3}{8} + \sqrt{\frac{3}{4}x + x^2} \right).$$

$$2. \, dy = \frac{3dx}{\sqrt{5x+9x^3}}. \quad \text{Ans. } y = l\left(x + \frac{5}{18} + \sqrt{\frac{4}{3}x + x^3}\right).$$

5°. Let it be required to integrate the expression,

$$dy = \frac{dx}{a^2 - x^2}.$$

Factoring, we have,

$$\frac{dx}{a^2 - x^2} = \frac{1}{2a} \left\{ \frac{dx}{a+x} + \frac{dx}{a-x} \right\}.$$

Integrating, we have,

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \left(l(a+x) - l(a-x) \right).$$

But the difference of the logarithms of two quantities is equal to the logarithm of their quotient; hence,

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} l \frac{a+x}{a-x} + C \dots \dots [33]$$

and in like manner,

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} l \frac{x-a}{x+a} + C \dots \dots [34]$$

EXAMPLES.

$$1. \, dy = \frac{dx}{9 - x^2}. \quad \text{Ans. } y = \frac{1}{6} l \frac{3+x}{3-x}.$$

$$2. \, dy = \frac{dx}{x^2 - 4}. \quad \text{Ans. } y = \frac{1}{4} l \frac{x-2}{x+2}.$$

3. Given $\frac{d^2y}{dx^2} = y$, to find the relation between y and x .

Multiplying both members by $2dy$,

$$\frac{2dy d^2y}{dx^2} = 2ydy, \text{ or } \frac{d(dy^2)}{dx^2} = 2ydy.$$

Integrating,

$$\frac{dy^2}{dx^2} = y^2 + C; \quad \therefore dx = \frac{dy}{\sqrt{y^2 + C}}.$$

Integrating by Formula 31,

$$x = l(y + \sqrt{y^2 + C}) + C'.$$

4. Given $\frac{d^2s}{dt^2} = -n^2s$, to find the relation between s

and t , t being the independent variable.

Multiplying by $2ds$,

$$\frac{2ds d^2s}{dt^2} = -2n^2s ds.$$

Integrating,

$$\frac{ds^2}{dt^2} = C - n^2s^2; \quad \therefore dt = \frac{ds}{\sqrt{C - n^2s^2}}.$$

Integrating by Formula (20),

$$t = \frac{1}{n} \sin^{-1} \frac{ns}{\sqrt{C}} + C'.$$

Rational and Entire Differentials.

65. An algebraic function is *rational and entire* when it contains no radical or fractional part that involves the variable. If the indicated operations be performed, every

rational and entire differential will be reduced to the form of a monomial, or polynomial differential, each term of which can be integrated by formulas (1) or (2). Thus, if the indicated operations be performed, in the expression,

$$dy = \left(x - \frac{1}{2}\right) \left(x^2 + \frac{3}{4}\right) x^3 dx,$$

we have,

$$dy = \frac{1}{8} \left(8x^6 - 4x^5 + 6x^4 - 3x^3 \right) dx.$$

Hence, by integration,

$$\begin{aligned} y &= \frac{1}{8} \left(\frac{8}{7} x^7 - \frac{2}{3} x^6 + \frac{6}{5} x^5 - \frac{3}{4} x^4 \right) \\ &= \left(\frac{1}{7} x^3 - \frac{1}{12} x^2 + \frac{3}{20} x - \frac{3}{32} \right) x^4. \end{aligned}$$

In like manner, any rational and entire differential may be integrated.

Rational Fractions.

66. A rational fraction, is a fraction whose terms are rational and entire. When a rational fraction is the differential of a function, and its numerator is of a higher degree than its denominator, it may, by the process of division, be separated into two differentials, one of which is rational and entire and the other a rational fraction, in which the numerator is of a lower degree than the denominator. The former may be integrated as in the last article; it remains to be shown that the latter can be integrated whenever the denominator can be resolved into binomial factors of the first degree with respect to the variable. The method

of integration consists in resolving the differential coefficients into partial fractions, by some of the known methods, then multiplying each by the differential of the variable and integrating the polynomial result. There are four cases, depending on the method of resolving the differential coefficient into partial fractions. Each case will be illustrated by examples.

First Case.—When the binomial factors of the denominator are unequal.

Let us have,

$$dy = \frac{(2x-5)dx}{x^3 - 6x^2 + 11x - 6} = \frac{2x-5}{(x-1)(x-2)(x-3)} dx.$$

Assume the identical equation,

$$\frac{2x-5}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \dots\dots (1)$$

Clearing of fractions.

$$2x-5 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \dots\dots (2)$$

In order to find A , B , and C , we might perform the operations indicated in (2), equate the coefficients of the like powers of x , and solve the resulting equations; but there is a simpler method in cases like this, depending on the fact that (2) is true for all values of x .

$$\text{Making } x = 1, \text{ we find, } -3 = 2A; \therefore A = -\frac{3}{2}.$$

$$\text{Making } x = 2, \text{ we find, } -1 = -B; \therefore B = 1.$$

$$\text{Making } x = 3, \text{ we find, } 1 = 2C; \therefore C = \frac{1}{2}.$$

Substituting these in (1), multiplying by dx , and integrating by Formula 30, we have,

$$\begin{aligned}\int \frac{(2x-5)dx}{(x-1)(x-2)(x-3)} &= -\frac{3}{2} \int \frac{dx}{x-1} + \int \frac{dx}{x-2} \\ &+ \frac{1}{2} \int \frac{dx}{x-3} = -\frac{3}{2} l(x-1) + l(x-2) + \frac{1}{2} l(x-3).\end{aligned}$$

Reducing the result to its simplest form, in accordance with the elementary principles of logarithms, we have, finally,

$$y = l \frac{(x-2)(x-3)^{\frac{1}{2}}}{(x-1)^{\frac{3}{2}}}.$$

EXAMPLES.

$$1. \quad dy = \frac{(5x+1)dx}{x^2+x-2} = \frac{(5x+1)dx}{(x-1)(x+2)}.$$

$$\text{Ans. } y = 2l(x-1) + 3l(x+2) = l[(x-1)^2 \times (x+2)^3].$$

$$2. \quad dy = \frac{(x-1)dx}{x^2+6x+8} = \frac{(x-1)dx}{(x+2)(x+4)}.$$

$$\text{Ans. } y = l \frac{(x+4)^{\frac{5}{2}}}{(x+2)^{\frac{3}{2}}}.$$

$$3. \quad dy = \frac{(2x+3)dx}{x^3+x^2-2x} = \frac{(2x+3)dx}{x(x-1)(x+2)}.$$

$$\text{Ans. } y = l \frac{(x-1)^{\frac{5}{2}}}{x^{\frac{3}{2}}(x+2)^{\frac{1}{2}}}.$$

$$4. \quad dy = \frac{(3x^2-1)dx}{x(x-1)(x+1)}.$$

$$\text{Ans. } y = l(x^3 - x).$$

Second Case.—When some of the binomial factors are equal.

In this case there are as many partial fractions as there are binomial factors in the denominator; but the denominators of those corresponding to factors of the form $(x - a)^m$, are respectively of the form $(x - a)^m$, $(x - a)^{m-1}$, $(x - a)^{m-2}$, etc., down to $x - a$.

Let us assume,

$$dy = \frac{(x^2 + x)dx}{(x - 2)^2(x - 1)};$$

assuming the identical equation,

$$\frac{x^2 + x}{(x - 2)^2(x - 1)} = \frac{A}{(x - 2)^2} + \frac{B}{x - 2} + \frac{C}{x - 1} \dots \dots (1)$$

and clearing of fractions, we have,

$$x^2 + x = A(x - 1) + B(x - 2)(x - 1) + C(x - 2)^2 \dots (2)$$

Here again we might find the values of A , B , and C , by the ordinary method of indeterminate coefficients; but it will be simpler to proceed as follows:

Making $x = 2$, we find, $A = 6$;

Making $x = 1$, we find, $C = 2$;

giving to A and C their values in (2), and then making $x = 0$, we find, $0 = -6 + 2B + 8 \therefore B = -1$.

Substituting in (1) and multiplying by dx , we have,

$$\frac{(x^2 + x)dx}{(x - 2)^2(x - 1)} = \frac{6dx}{(x - 2)^2} - \frac{dx}{x - 2} + \frac{2dx}{x - 1}.$$

The first partial fraction can be integrated by Formula 29, and the others by Formula 30; hence,

$$y = -\frac{6}{x-2} - l(x-2) + 2l(x-1) = -\frac{6}{x-2} + l\frac{(x-1)^2}{x-2}.$$

EXAMPLES.

$$1. \quad dy = \frac{(x^2 - 2)dx}{x^3(x-1)}.$$

In this example the equal binomial factors are of the form $(x-0)$ or x , and the assumed identical equation is,

$$\frac{x^2 - 2}{x^3(x-1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-1} \dots \dots (1)$$

Clearing of fractions, and performing indicated operations, we have,

$$x^2 - 2 = Ax - A + Bx^2 - Bx + Cx^3 - Cx^2 + Dx^3.$$

Equating the coefficients of like powers in the two members, and solving the resulting group, we have, $A = 2$, $B = 2$, $C = 1$, and $D = -1$; substituting in (1), and proceeding as before, we have,

$$y = -\frac{2x+1}{x^2} + l\frac{x}{x-1}.$$

$$2. \quad dy = \frac{x^2 dx}{(x+2)^2(x+1)}. \quad \text{Ans. } y = \frac{4}{x+2} + l(x+1).$$

$$3. \quad dy = \frac{x dx}{(x+2)(x+3)^2}. \quad \text{Ans. } y = -\frac{3}{x+3} + l\left(\frac{x+3}{x+2}\right)^2.$$

Third Case.—When some of the factors are imaginary, but unequal.

In this case the product of each pair of imaginary factors is of the form $(x - a)^2 + b^2$; instead of assuming a partial fraction for each imaginary factor, we assume for each pair of such factors a fraction of the form,

$$\frac{A + Bx}{(x - a)^2 + b^2}.$$

$$\text{Let } dy = \frac{x dx}{(x + 1)(x^2 + 1)}.$$

Assume the identical equation,

$$\frac{x}{(x + 1)(x^2 + 1)} = \frac{A + Bx}{x^2 + 1} + \frac{C}{x + 1} \dots \dots (1)$$

Clearing of fractions, and reducing,

$$x = Bx^2 + (A + B)x + A + Cx^2 + C \dots \dots (2)$$

Equating the coefficients of the like powers of x in the two members, and solving the resulting equations, we have,

$$A = \frac{1}{2}, \quad B = \frac{1}{2}, \quad \text{and} \quad C = -\frac{1}{2}.$$

Substituting in (1), and multiplying by dx , we have,

$$\begin{aligned} \frac{x dx}{(x + 1)(x^2 + 1)} &= \frac{1}{2} \frac{(1 + x) dx}{x^2 + 1} - \frac{1}{2} \frac{dx}{x + 1} \\ &= \frac{1}{2} \left(\frac{dx}{x^2 + 1} + \frac{x dx}{x^2 + 1} - \frac{dx}{x + 1} \right). \end{aligned}$$

Integrating by Formulas (14) and (30), and reducing, we have,

$$y = \frac{1}{2} \tan^{-1} x + \frac{1}{2} \sqrt{\frac{x^2 + 1}{x + 1}}.$$

Fourth Case.—When some of the binomial factors are imaginary and equal.

In this case, the denominator will contain one or more factors of the form $[(x - a)^2 + b^2]^n$; in determining the partial fractions, we combine the methods used in the second and third cases.

Let us assume,

$$dy = \frac{x^2 dx}{(x^2 - 2x + 2)^2}.$$

Assume the identical equation,

$$\frac{x^2}{(x^2 - 2x + 2)^2} = \frac{Ax + B}{(x^2 - 2x + 2)^2} + \frac{Cx + D}{x^2 - 2x + 2} \dots \dots (1)$$

Clearing of fractions, and proceeding as before, we find,

$$A = 2, B = -4, C = 1, \text{ and } D = 2.$$

Substituting these in (1), multiplying by dx , and separating the fractions, we have,

$$dy = \frac{2x dx}{(x^2 - 2x + 2)^2} - \frac{4 dx}{(x^2 - 2x + 2)^2} + \frac{x dx}{x^2 - 2x + 2} + \frac{2 dx}{x^2 - 2x + 2} \dots \dots (2)$$

Making $(x - 1)^2 + 1 = z^2 + 1$, whence $x - 1 = z$, and $dx = dz$, we have,

$$dy = \frac{2(z + 1) dz}{(z^2 + 1)^2} - \frac{4 dz}{(z^2 + 1)^2} + \frac{(z + 1) dz}{z^2 + 1} + \frac{2 dz}{z^2 + 1},$$

or,

$$dy = \frac{2z dz}{(z^2 + 1)^2} - \frac{2 dz}{(z^2 + 1)^2} + \frac{z dz}{z^2 + 1} + \frac{3 dz}{z^2 + 1}.$$

The *first*, *third*, and *fourth* terms can be integrated by known formulas; the *second* is a particular case of the form $\frac{dz}{(z^2 + 1)^n}$, which can be integrated by the aid of Formula *D*, yet to be deduced.

Integrating the *first*, *third*, and *fourth*, and indicating the integration of the *second*, we have,

$$v = -\frac{1}{(z^2 + 1)} - 2 \int \frac{dz}{(z^2 + 1)^2} + \frac{1}{2} \log(z^2 + 1) + 3 \tan^{-1} z.$$

From what precedes we infer that all rational differentials are integrable; consequently, all differentials that can be made rational in terms of a new variable are also integrable.

Integration by Substitution, and Rationalization.

67. An irrational differential may sometimes be made rational, by substituting for the variable some function of an auxiliary variable; when this can be done, the integration may be effected by the methods of Articles 65 and 66. When the differential cannot be rationalized in terms of an auxiliary variable, it may sometimes be reduced to one of the elementary forms, and then integrated. The method of proceeding is best illustrated by examples.

When the only Irrational Parts are Monomial.

68. When the only irrational parts are monomial, a differential can be made rational by substituting for the primitive variable a new variable raised to a power whose

exponent is the least common multiple of all the indices in the expression.

$$\text{Let} \quad dy = \frac{(x^{\frac{1}{2}} - x^{\frac{2}{3}})dx}{1 + x^{\frac{1}{3}}} \dots\dots (1)$$

The least common multiple of the indices is 6; making $x = z^6$, we have, $x^{\frac{1}{2}} = z^3$, $x^{\frac{2}{3}} = z^4$, $x^{\frac{1}{3}} = z^2$, and $dx = 6z^5 dz$: these substituted in (1), give,

$$dy = \frac{(z^3 - z^4)}{1 + z^2} 6z^5 dz = -6 \frac{z^9 - z^8}{z^2 + 1} dz.$$

Performing the division indicated, we have,

$$dy = -6 \left(z^7 - z^6 - z^5 + z^4 + z^3 - z^2 - z + 1 + \frac{z}{z^2 + 1} - \frac{1}{z^2 + 1} \right) dz.$$

Integrating by known methods, and replacing z by its value, $x^{\frac{1}{6}}$, we have, finally,

$$y = -\frac{3}{4}x^{\frac{4}{3}} + \frac{6}{7}x^{\frac{7}{6}} + x - \frac{6}{5}x^{\frac{5}{6}} - \frac{3}{2}x^{\frac{2}{3}} \\ + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} - 3l(1 + x^{\frac{1}{3}}) + 6\tan^{-1}x^{\frac{1}{6}}.$$

When the only irrational parts are fractional powers of a binomial of the first degree, the differential can be rationalized by the same rule, and consequently can be integrated.

Let us take the expression,

$$dy = (x + \sqrt{x+2} + \sqrt[3]{x+2}) dx \dots\dots (2)$$

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Assume,

$$x + 2 = z^6, \text{ whence, } x = z^6 - 2, \text{ and } dx = 6z^5 dz.$$

Substituting in (2), we have,

$$dy = (z^6 - 2 + z^3 + z^3)6z^5 dz = 6(z^{11} - 2z^5 + z^8 + z^6)dz.$$

Integrating, and substituting for z its value, $(x + 2)^{\frac{1}{6}}$, we have,

$$y = \frac{1}{2}(x + 2)^{\frac{7}{6}} - 2(x + 2)^{\frac{1}{6}} + \frac{2}{3}(x + 2)^{\frac{4}{3}} + \frac{3}{4}(x + 2)^{\frac{5}{6}}.$$

Binomial Differentials.

69. Every binomial differential can be reduced to the form,

$$dy = A(a + bx^n)^{\frac{r}{s}} x^{m-1} dx \dots \dots (1)$$

in which m , n , r , and s are whole numbers, and n positive.

Thus, the expression, $(x^{-\frac{1}{2}} + 2x^{-\frac{1}{4}})^{-\frac{2}{3}} x^{\frac{1}{2}} dx$, can be reduced to the form $(1 + 2x^{\frac{1}{4}})^{-\frac{2}{3}} x^{\frac{1}{2}} dx$ by simply removing the factor $x^{-\frac{1}{2}}$ from under the radical sign. If, in this result, we make $x = z^4$, whence, $dx = 4z^3 dz$, it will become, $4(1 + 2z)^{-\frac{2}{3}} z^5 dz$, which is of the required form. In like manner, any similar expression may be reduced to that form. The constant factor, A , may be omitted during integration, and the exponent of the parenthesis, $\frac{r}{s}$, may be represented by a single letter p , as is done in Article 70. After integration, the factor may be replaced, and the value of $\frac{r}{s}$ may be substituted for p .

FIRST CRITERION.

Form (1) may be integrated by some of the methods explained in Articles 65 and 66, when $\frac{r}{s}$ is a whole number.

SECOND CRITERION.

Form (1) can be integrated when $\frac{m}{n}$ is a whole number.

For let us assume $(a + bx^n) = z^s$, or, $z = (a + bx^n)^{\frac{1}{s}}$; we have, by solution and differentiation,

$$x = \left(\frac{z^s - a}{b}\right)^{\frac{1}{n}}; \quad x^m = \left(\frac{z^s - a}{b}\right)^{\frac{m}{n}};$$

and,

$$x^{m-1}dx = \frac{s}{bn} \left(\frac{z^s - a}{b}\right)^{\frac{m}{n}-1} z^{s-1}dz.$$

Substituting in (1), we find,

$$\left(a + bx^n\right)^{\frac{r}{s}} x^{m-1} dx = \frac{s}{bn} \left(\frac{z^s - a}{b}\right)^{\frac{m}{n}-1} z^{r+s-1} dz \dots (2)$$

Which expression is integrable, as was shown above, when $\frac{m}{n}$ is a whole number.

THIRD CRITERION.

Form (1) can be integrated when $\frac{m}{n} + \frac{r}{s}$, is a whole number.

For let us assume,

$$a + bx^n = z^s x^n, \text{ or, } z = \left(\frac{a + bx^n}{x^n}\right)^{\frac{1}{s}}.$$

Solving and differentiating, we have,

$$x = \left(\frac{a}{z^s - b} \right)^{\frac{1}{n}} = a^{\frac{1}{n}} (z^s - b)^{-\frac{1}{n}}; \quad x^m = a^{\frac{m}{n}} (z^s - b)^{-\frac{m}{n}};$$

and,

$$x^{m-1} dx = -\frac{s}{n} a^{\frac{m}{n}} (z^s - b)^{-\frac{m}{n}-1} z^{s-1} dz.$$

Substituting in (1), and reducing, we find,

$$(a + bx^n)^{\frac{r}{n}} x^{m-1} dx = -\frac{s}{n} a^{\frac{m}{n} + \frac{r}{s}} (z^s - b)^{-\frac{m}{n} - \frac{r}{s} - 1} z^{r+s-1} dz$$

. (3)

Which can be integrated, as was shown above, when $\frac{m}{n} + \frac{r}{s}$ is a whole number.

EXAMPLES.

1. $dy = (1 + x^2)^{\frac{1}{2}} x^3 dx$ (4)

Here, $\frac{m}{n} = \frac{4}{2} = 2$; hence, the second criterion is satisfied.

Comparing (2) with (4), we find,

$$a = 1, b = 1, n = 2, m = 4, r = 1, s = 2, \text{ and } z = (1 + x^2)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned} \int (1 + x^2)^{\frac{1}{2}} x^3 dx &= \int (z^2 - 1) z^3 dz = \frac{z^5}{5} - \frac{z^3}{3} \\ &= \frac{(1 + x^2)^{\frac{5}{2}}}{5} - \frac{(1 + x^2)^{\frac{3}{2}}}{3} = \frac{(3x^2 - 2)(1 + x^2)^{\frac{3}{2}}}{15}. \end{aligned}$$

$$2. \, dy = \frac{dx}{x^4(1+x^2)^{\frac{1}{2}}} = dx(1+x^2)^{-\frac{1}{2}}x^{-4} \dots \dots (5)$$

Here, $\frac{m}{n} + \frac{r}{s} = -\frac{3}{2} - \frac{1}{2} = -2$; hence, the third criterion is satisfied.

Comparing (3) and (5), we find,

$a = 1$, $b = 1$, $n = 2$, $m = -3$, $r = -1$, $s = 2$, and

$$z = \frac{\sqrt{1+x^2}}{x}.$$

Hence,

$$\begin{aligned} \int \frac{dx}{x^4(1+x^2)^{\frac{1}{2}}} &= -\int (z^2 - 1)dz = z - \frac{z^3}{3} = \frac{3z - z^3}{3} \\ &= \frac{(2x^2 - 1)(1+x^2)^{\frac{1}{2}}}{3x^3}. \end{aligned}$$

Integration by Successive Reduction.

70. When an integral can be made to depend upon a simpler integral of the same form, it is evident that by successive repetitions of the process, we may ultimately arrive at a form that can be integrated by one of the fundamental formulas. This method of integration is called *the method by successive reduction*, and is effected by *formulas of reduction*, some of which it is now proposed to investigate.

1°. If in the expression for a binomial differential, we omit the factor A , represent the exponent of the parenthesis by p , and factor the result, we have,

$$(a + bx^n)^p x^{m-1} dx = x^{m-n} (a + bx^n)^p x^{n-1} dx \dots \dots (1)$$

Let $u = x^{m-n}$, and $dv = (a + bx^n)^p x^{n-1} dx$.

Differentiating the first, and integrating the second by Formula (29), we have,

$$du = (m - n)x^{m-n-1} dx; \text{ and } v = \frac{(a + bx^n)^{p+1}}{bn(p+1)};$$

But, $(a + bx^n)^{p+1} = (a + bx^n)^p (a + bx^n)$;

$$\therefore v = \frac{a(a + bx^n)^p}{bn(p+1)} + \frac{(a + bx^n)^p x^n}{n(p+1)}.$$

Substituting in Formula (28), we have,

$$\begin{aligned} \int (a + bx^n)^p x^{m-1} dx &= \frac{(a + bx^n)^{p+1} x^{m-n}}{bn(p+1)} \\ &\quad - \frac{a(m-n)}{bn(p+1)} \int (a + bx^n)^p x^{m-n-1} dx \\ &\quad - \frac{(m-n)}{n(p+1)} \int (a + bx^n)^p x^{n-1} dx. \end{aligned}$$

Transposing the last term to the first member, and reducing, we have,

$$\begin{aligned} \frac{pn+m}{n(p+1)} \int (a + bx^n)^p x^{m-1} dx &= \frac{(a + bx^n)^{p+1} x^{m-n}}{bn(p+1)} \\ &\quad - \frac{a(m-n)}{bn(p+1)} \int (a + bx^n)^p x^{m-n-1} dx. \end{aligned}$$

Dividing by the coefficient in the first member, we have,

$$\begin{aligned} \int (a + bx^n)^p x^{m-1} dx &= \frac{(a + bx^n)^{p+1} x^{m-n}}{b(pn+m)} \\ &\quad - \frac{a(m-n)}{b(pn+m)} \int (a + bx^n)^p x^{m-n-1} dx \dots \dots [A] \end{aligned}$$

Formula *A* enables us to reduce the exponent of the variable without the parenthesis, by the exponent of the variable within the parenthesis, at each application. It fails when $pn + m = 0$; but in this case, the third criterion, (Art. 69), is satisfied, and the expression may be integrated by a previous method.

2°. The expression $(a + bx^n)^p$ may be placed under the form,

$$(a + bx^n)^{p-1}(a + bx^n),$$

$$\text{or, } a(a + bx^n)^{p-1} + b(a + bx^n)^{p-1}x^n;$$

hence, we may write the equation,

$$\int (a + bx^n)^p x^{m-1} dx$$

$$= a \int (a + bx^n)^{p-1} x^{m-1} dx + b \int (a + bx^n)^{p-1} x^{m+n-1} dx$$

$$\dots\dots (2)$$

The last term of this equation can be reduced by Formula *A*. Replacing m by $m + n$, p by $p - 1$, and multiplying by b , that formula becomes,

$$b \int (a + bx^n)^{p-1} x^{m+n-1} dx$$

$$= \frac{(a + bx^n)^p x^m}{pn + m} - \frac{am}{pn + m} \int (a + bx^n)^{p-1} x^{m-1} dx.$$

Substituting in (2), and uniting similar terms, we have,

$$\int (a + bx^n)^p x^{m-1} dx$$

$$= \frac{(a + bx^n)^p x^m}{pn + m} + \frac{pna}{pn + m} \int (a + bx^n)^{p-1} x^{m-1} dx \dots [B]$$

Formula *B* enables us to reduce the exponent of the parenthesis by 1 at each application. It fails in the same case as Formula *A*. When *m* and *p* are negative, we endeavor to increase instead of diminishing them. For this purpose we reverse Formulas *A* and *B*.

3°. Reversing Formula *A*, and reducing, we have,

$$\int (a + bx^n)^p x^{m-n-1} dx \\ = \frac{(a + bx^n)^{p+1} x^{m-n}}{a(m-n)} - \frac{b(pn+m)}{a(m-n)} \int (a + bx^n)^p x^{m-1} dx.$$

Replacing *m* by $-m + n$, we have,

$$\int (a + bx^n)^p x^{-m-1} dx = - \frac{(a + bx^n)^{p+1} x^{-m}}{am} \\ + \frac{b(pn-m+n)}{am} \int (a + bx^n)^p x^{-m+n-1} dx \dots [C]$$

4°. Reversing Formula *B*, and reducing, we have,

$$\int (a + bx^n)^{p-1} x^{m-1} dx \\ = - \frac{(a + bx^n)^p x^m}{pna} + \frac{pn+m}{pna} \int (a + bx^n)^p x^{m-1} dx.$$

In this, substituting $-p + 1$ for *p*, we have,

$$\int (a + bx^n)^{-p} x^{m-1} dx = \frac{(a + bx^n)^{-p+1} x^m}{an(p-1)} \\ - \frac{n+n-np}{an(p-1)} \int (a + bx^n)^{-p+1} x^{m-1} dx \dots [D]$$

The mode of applying Formulas *A*, *B*, *C*, and *D*, will be illustrated by practical examples.

1. Let it be required to integrate the expression,

$$(r^2 - x^2)^{\frac{1}{2}} x^2 dx.$$

In Formula *A*, making

$$a = r^2, b = -1, n = 2, p = \frac{1}{2}, \text{ and } m = 3,$$

we find,

$$\int (r^2 - x^2)^{\frac{1}{2}} x^2 dx = -\frac{(r^2 - x^2)^{\frac{1}{2}} x}{4} + \frac{r^2}{4} \int (r^2 - x^2)^{\frac{1}{2}} dx$$

. (3)

In Formula *B*, making

$$a = r^2, b = -1, n = 2, p = \frac{1}{2}, \text{ and } m = 1,$$

we find,

$$\int (r^2 - x^2)^{\frac{1}{2}} dx = \frac{(r^2 - x^2)^{\frac{1}{2}} x}{2} + \frac{r^2}{2} \int (r^2 - x^2)^{-\frac{1}{2}} dx$$

. (4)

$$\text{But } \int (r^2 - x^2)^{-\frac{1}{2}} dx = \int \frac{dx}{\sqrt{r^2 - x^2}} = \sin^{-1} \frac{x}{r}.$$

Substituting this in (4), and that result in (3), we have, finally,

$$\begin{aligned} \int (r^2 - x^2)^{\frac{1}{2}} x^2 dx = & -\frac{(r^2 - x^2)^{\frac{3}{2}} x}{4} \\ & + \frac{r^2 (r^2 - x^2)^{\frac{1}{2}} x}{8} + \frac{r^4}{8} \sin^{-1} \frac{x}{r}. \end{aligned}$$

2. Let $dy = \frac{dx}{x^3(x^2 - a^2)^{\frac{1}{2}}} = (-a^2 + x^2)^{-\frac{1}{2}}x^{-3}dx$.

In Formula *C*, making $a = -a^2$, $b = 1$, $n = 2$, $p = -\frac{1}{2}$, and $m = 2$, we have,

$$\int (-a^2 + x^2)^{-\frac{1}{2}}x^{-3}dx = + \frac{(x^2 - a^2)^{\frac{1}{2}}x^{-2}}{2a^2} \\ + \frac{1}{2a^2} \int (-a^2 + x^2)^{-\frac{1}{2}}x^{-1}dx \dots \dots (5)$$

But by Formula (24),

$$\int (-a^2 + x^2)^{-\frac{1}{2}}x^{-1}dx = \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}.$$

Substituting in (5), we have, for the value of y ,

$$\int \frac{dx}{x^3(x^2 - a^2)^{\frac{1}{2}}} = \frac{(x^2 - a^2)^{\frac{1}{2}}}{2a^2x^2} + \frac{1}{2a^2} \sec^{-1} \frac{x}{a}.$$

3. Let $dy = \frac{dz}{(z^2 + 1)^2} = (z^2 + 1)^{-2}dz$.

In Formula *D*, making $x = z$, $a = 1$, $b = 1$, $n = 2$, $p = 2$, and $m = 1$, we have,

$$\int (z^2 + 1)^{-2} dz = \frac{(z^2 + 1)^{-1}z}{2} + \frac{1}{2} \int (z^2 + 1)^{-1} dz.$$

But,

$$\int (z^2 + 1)^{-1} dz = \int \frac{dz}{1 + z^2} = \tan^{-1} z.$$

Hence, we have, for the value of y ,

$$\int \frac{dz}{(z^2 + 1)^2} = \frac{z}{2(z^2 + 1)} + \frac{1}{2} \tan^{-1} z.$$

This is the method of integration referred to under the Fourth Case of rational fractions. By a continued application of Formula D , any expression of the form,

$\frac{dz}{(a^2 + z^2)^n}$, can be reduced to an integrable form.

The following additional examples can be reduced to integrable forms by the application of Formulas A , B , C , and D .

$$1. \quad dy = \frac{x^4 dx}{(a^2 - x^2)^{\frac{3}{2}}}.$$

$$Ans. \quad y = -\frac{x^3(a^2 - x^2)^{\frac{1}{2}}}{4} - \frac{3a^2 x}{8(a^2 - x^2)^{\frac{1}{2}}} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a}.$$

$$5. \quad dy = (a^2 - x^2)^{\frac{1}{2}} dx.$$

$$Ans. \quad y = \frac{1}{2} x(a^2 - x^2)^{\frac{1}{2}} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

$$6. \quad dy = (1 + x^2)^{\frac{3}{2}} x^3 dx. \quad Ans. \quad y = \frac{5x^5 - 2}{35} (1 + x^2)^{\frac{5}{2}}.$$

$$7. \quad dy = \frac{x^3 dx}{(1 + x^2)^{\frac{3}{2}}}. \quad Ans. \quad y = \frac{x^2 + 2}{\sqrt{1 + x^2}}.$$

$$8. \quad dy = \frac{x^3 dx}{(1 + x^2)^{\frac{5}{2}}}. \quad Ans. \quad y = -\frac{3x^2 + 2}{3(1 + x^2)^{\frac{3}{2}}}.$$

$$9. \quad dy = \frac{x^4 dx}{(1 - x^2)^{\frac{3}{2}}}. \quad Ans. \quad y = -\frac{x^3 - 3x}{2\sqrt{1 - x^2}} - \frac{3}{2} \sin^{-1} x.$$

$$10. \, dy = \frac{dx}{x^4(1-x^2)^{\frac{1}{2}}}. \quad \text{Ans. } y = -\frac{2x^2+1}{3x^3}(1-x^2)^{\frac{1}{2}}.$$

5°. Let it be required to simplify the expression,

$$\frac{x^n dx}{\sqrt{2ax-x^2}}.$$

Changing the form and removing the factor x from under the radical sign, we have,

$$(2a-x)^{-\frac{1}{2}} x^{n-\frac{1}{2}} dx;$$

in formula A , making,

$$a = 2a, \, b = -1, \, n = 1, \, p = -\frac{1}{2}, \text{ and } m = n + \frac{1}{2},$$

and reducing, we have,

$$b(pn+m) = -n, \text{ and } a(m-n) = 2a(n-\frac{1}{2}) = a(2n-1);$$

hence,

$$\begin{aligned} \int \frac{x^n dx}{\sqrt{2ax-x^2}} &= -\frac{x^{n-1} \sqrt{2ax-x^2}}{n} \\ &+ \frac{(2n-1)a}{n} \int \frac{x^{n-1} dx}{\sqrt{2ax-x^2}} \dots \dots [E] \end{aligned}$$

Formula E enables us to reduce the exponent n by 1, at each application; if n is entire and positive, by a continued application of the formula we ultimately arrive at

the form, $\int \frac{dx}{\sqrt{2ax-x^2}}$, which is equal to $\text{versin}^{-1} \frac{x}{a}$.

EXAMPLES.

$$1. \quad dy = \frac{x dx}{\sqrt{2ax - x^2}}.$$

$$Ans. \quad y = -\sqrt{2ax - x^2} + a \operatorname{versin}^{-1} \frac{x}{a}.$$

$$2. \quad dy = \frac{x^2 dx}{\sqrt{2x - x^2}}.$$

$$Ans. \quad y = -\frac{x+3}{2} \sqrt{2x - x^2} + \frac{3}{2} \operatorname{versin}^{-1} x.$$

$$3. \quad dy = \frac{x^3 dx}{\sqrt{2x - x^2}}.$$

$$Ans. \quad y = -\frac{2x^2 + 5x + 15}{6} \sqrt{2x - x^2} + \frac{5}{2} \operatorname{versin}^{-1} x.$$

Certain Trinomial Differentials.

71. Every trinomial differential that can be reduced to the form

$$(a + bx \pm x^2)^{\frac{p}{2}} x^m dx \dots \dots (1)$$

can be made rational in terms of an auxiliary variable, and consequently can be integrated, when p and m are whole numbers.

When p is *even*, the expression is already rational; when p is *odd*, say of the form $2n + 1$, the expression becomes,

$$(a + bx \pm x^2)^n (a + bx \pm x^2)^{\frac{1}{2}} x^m dx \dots \dots (2)$$

in which the only irrational part is $\sqrt{a + bx \pm x^2}$. There may be two cases; *first*, when x^2 is positive, and *secondly*, when x^2 is negative.

1°. When x^2 is positive.

Assume,

$$\sqrt{a + bx + x^2} = z - x; \therefore a + bx + x^2 = z^2 - 2zx + x^2;$$

striking out the common term x^2 , and solving, we have,

$$x = \frac{z^2 - a}{2z + b}; \therefore dx = \frac{2(z^2 + bz + a)}{(2z + b)^2} dz,$$

and

$$\sqrt{a + bx + x^2} = \frac{z^2 + bz + a}{2z + b} \dots\dots (3)$$

Substituting these in (2), the result will be rational.

EXAMPLES.

$$1. dy = \frac{dx}{\sqrt{1 + x + x^2}} = (1 + x + x^2)^{-\frac{1}{2}} dx \dots\dots (4)$$

Comparing (4) with (1), we find, $a = 1$, and $b = 1$.
Hence, from (3), we have,

$$\sqrt{1 + x + x^2} = \frac{z^2 + z + 1}{2z + 1}, \text{ and } dx = \frac{2(z^2 + z + 1)}{(2z + 1)^2} dz.$$

Substituting in (4), and making $z = x + \sqrt{1 + x + x^2}$, we have,

$$dy = \frac{2dz}{2z + 1};$$

$$\therefore y = l(2z + 1) = l(2x + 1 + 2\sqrt{1 + x + x^2}).$$

$$2. dy = \frac{dx}{\sqrt{x^2 - x - 1}}.$$

$$Ans. y = l(2x - 1 + 2\sqrt{x^2 - x - 1}).$$

$$3. \, dy = \frac{dx}{x\sqrt{1+x+x^2}}.$$

$$\text{Ans. } y = l\left(\frac{3x}{2+x+2\sqrt{1+x+x^2}}\right).$$

2°. When x^2 is negative.

Let α and β be taken so as to satisfy the equation,

$$a + bx - x^2 = (x - \alpha)(\beta - x) \dots\dots (5)$$

Assume

$$\sqrt{a + bx - x^2} = \sqrt{(x - \alpha)(\beta - x)} = (x - \alpha)z;$$

squaring and reducing, we have,

$$\beta - x = (x - \alpha)z^2; \text{ or, } z = \sqrt{\frac{\beta - x}{x - \alpha}};$$

hence,

$$x = \frac{\beta + \alpha z^2}{1 + z^2}; \, dx = -\frac{2(\beta - \alpha)zdz}{(1 + z^2)^2};$$

and

$$\sqrt{a + bx - x^2} = \frac{(\beta - \alpha)z}{1 + z^2} \dots\dots (6)$$

These values substituted in (1), will make it rational.

EXAMPLE.

$$\text{Let } dy = \frac{dx}{\sqrt{2 - x - x^2}} = \frac{dx}{\sqrt{(x + 2)(1 - x)}};$$

$$\therefore \alpha = -2, \text{ and } \beta = 1 \dots\dots (7)$$

From (6), we have,

$$dx = -\frac{6zdz}{(1+z^2)^2}, \text{ and } \sqrt{2-x-x^2} = \frac{3z}{(1+z^2)};$$

$$\text{also } z = \sqrt{\frac{1-x}{x+2}}.$$

Substituting in (7), we find,

$$dy = -\frac{2dz}{1+z^2}; \therefore y = -2\tan^{-1}z = -2\tan^{-1}\sqrt{\frac{1-x}{x+2}}.$$

Integration by Series.

72. It is often convenient to develop a differential into a series, arranged according to the ascending powers of the variable, and then to integrate each term separately. This method sometimes enables us to find an approximate value for an integral, which cannot be found in any other manner; it also enables us to deduce many useful formulas.

EXAMPLES.

$$1. \quad dy = x^2(1-x^2)^{\frac{1}{2}}dx.$$

Developing $(1-x^2)^{\frac{1}{2}}$ by the binomial formula, and multiplying each term by x^2dx , we have,

$$dy = (x^2 - \frac{x^4}{2} - \frac{x^6}{8} - \frac{x^8}{16} - \text{etc.})dx.$$

$$\therefore y = \frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{56} - \frac{x^9}{144} - \text{etc.}$$

$$2. \quad dy = \frac{dx}{1+x^2} = dx(1+x^2)^{-1}.$$

By Formula (14) y is equal to $\tan^{-1}x$. Developing by the binomial formula, we have,

$$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \text{etc.}$$

Substituting, and integrating, we have the Formula,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \text{etc.}$$

$$3. \quad dy = \frac{dx}{1+x} = dx(1+x)^{-1}$$

$$= (1 - x + x^2 - x^3 + \text{etc.})dx$$

$$\therefore y = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}$$

$$4. \quad dy = \frac{dx}{\sqrt{1-x^2}} = dx(1-x^2)^{-\frac{1}{2}}.$$

$$\text{Ans. } y = \sin^{-1}x = x + \frac{x^3}{1.2.3} + \frac{3x^5}{2.4.5} + \frac{3.5x^7}{2.4.6.7} + \text{etc.}$$

Integration of Transcendental Differentials.

73. A transcendental differential is one that is expressed in terms of some transcendental quantity. There are three principal classes; viz., *logarithmic*, *exponential*, and *circular*.

Logarithmic Differentials.

74. A large number of cases come under the general form,

$$dy = x^{m-1}(lx)^n dx \dots\dots (1)$$

Assume,

$$u = (lx)^n, \text{ and } dv = x^{m-1} dx;$$

$$\therefore du = \frac{n(lx)^{n-1} dx}{x}, \text{ and } v = \frac{x^m}{m};$$

substituting in Formula (28), and reducing, we have,

$$y = \int x^{m-1} (lx)^n dx = \frac{x^m (lx)^n}{m} - \frac{n}{m} \int x^{m-1} (lx)^{n-1} dx \dots [F]$$

Formula F is a formula of reduction; it reduces the exponent of (lx) by 1 at each application.

EXAMPLES.

1. $dy = x(lx)^3 dx.$

$$Ans. y = \frac{x^5}{2} (lx)^3 - \frac{3x^4}{4} (lx)^2 + \frac{3x^3}{4} (lx) - \frac{3x^2}{8}.$$

2. $dy = x^3 (lx) dx.$

$$Ans. y = \frac{x^4 (lx)}{4} - \frac{x^4}{16}.$$

3. $dy = x^3 (lx)^3 dx.$

$$Ans. y = \frac{x^4}{4} \left((lx)^3 - \frac{3}{4} (lx)^2 + \frac{3}{8} (lx) - \frac{3}{32} \right).$$

Reversing Formula F , and reducing, we have,

$$\int x^{m-1} (lx)^{n-1} dx = \frac{x^m (lx)^n}{n} - \frac{m}{n} \int x^{m-1} (lx)^n dx.$$

Replacing $n - 1$ by $-n$, and reducing, we have,

$$\int \frac{x^{m-1} dx}{(lx)^n} = -\frac{x^m}{(n-1)(lx)^{n-1}} + \frac{m}{n-1} \int \frac{x^{m-1} dx}{(lx)^{n-1}} \dots [G]$$

The continued application of formula G gives as a final integral, the expression $\int \frac{x^{m-1} dx}{(lx)}$, which can be integrated by series. It fails when $n = 1$. Let $m = 0$, and $n = 1$; we then have,

$$\int \frac{dx}{xlx} = \int \frac{d(lx)}{lx} = l(lx) = l^2 x.$$

Exponential Differentials.

75. Let it be required to integrate an expression of the form,

$$dy = x^m a^x dx \dots \dots (1)$$

Assume,

$$u = x^m, \text{ and } dv = a^x dx; \therefore du = mx^{m-1} dx, \text{ and } v = \frac{a^x}{la}.$$

Substituting in Formula (28),

$$\int x^m a^x dx = \frac{x^m a^x}{la} - \frac{m}{la} \int x^{m-1} a^x dx \dots \dots [H]$$

When $m < 1$, we have,

$$\int \frac{a^x dx}{x^m} = -\frac{a^x}{(m-1)x^{m-1}} + \frac{la}{m-1} \int \frac{a^x dx}{x^{m-1}} \dots \dots [I]$$

EXAMPLES.

$$1. \, dy = xa^x dx. \qquad \text{Ans. } y = \frac{a^x}{la} \left(x - \frac{1}{la} \right).$$

$$2. \, dy = x^3 a^x dx. \qquad \text{Ans. } y = \frac{a^x}{la} \left(x^3 - \frac{3x^2}{la} + \frac{6x}{(la)^2} - \frac{6}{(la)^3} \right).$$

$$3. \, dy = \frac{e^x dx}{x^2}. \qquad \text{Ans. } y = -\frac{e^x}{x} + \int \frac{e^x dx}{x}.$$

But by means of McLaurin's Formula, we find,

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{etc.};$$

hence,

$$\begin{aligned} \int \frac{e^x dx}{x} &= \int \left(\frac{dx}{x} + dx + \frac{xdx}{1.2} + \text{etc.} \right) \\ &= lx + x + \frac{x^2}{1.2.2} + \frac{x^3}{1.2.3.3} + \text{etc.}, \end{aligned}$$

which, being substituted above, gives,

$$y = -\frac{e^x}{x} + lx + x + \frac{x^2}{1.2.2} + \frac{x^3}{1.2.3.3} + \text{etc.}$$

$$4. \, dy = x^2 e^x dx. \qquad \text{Ans. } y = e^x (x^2 - 2x + 2).$$

Circular Differentials.

76. Circular differentials may be integrated by the methods of *transformation* and *successive reduction*, or they may be reduced to algebraic forms by making $\sin x = z$.

EXAMPLES.

1. Let $dy = \frac{dx}{\sin x}$.

From trigonometry, we have,

$$\sin x = 2\sin\frac{1}{2}x\cos\frac{1}{2}x = 2\tan\frac{1}{2}x\cos^2\frac{1}{2}x,$$

or,

$$\frac{dx}{\sin x} = \frac{dx}{2\tan\frac{1}{2}x\cos^2\frac{1}{2}x} = \frac{d(\tan\frac{1}{2}x)}{\tan\frac{1}{2}x}.$$

Hence,

$$\int \frac{dx}{\sin x} = l(\tan\frac{1}{2}x) \dots\dots (1)$$

2. Let $dy = \frac{dx}{\cos x}$.

If we make x equal to $90^\circ - x$ in the preceding formula, and reduce, we have,

$$\int \frac{dx}{\cos x} = -l[\tan(45^\circ - \frac{1}{2}x)] \dots\dots (2)$$

3. Let $dy = \frac{dx}{\tan x}$.

From trigonometry, we have,

$$\tan x = \frac{\sin x}{\cos x}; \text{ or, } \frac{dx}{\tan x} = \frac{dx\cos x}{\sin x} = \frac{d(\sin x)}{\sin x}.$$

Hence,

$$\int \frac{dx}{\tan x} = l(\sin x) \dots\dots (3)$$

4. Let $dy = \frac{dx}{\cot x}$.

Making x equal to $90^\circ - x$ in (3), and reducing, we have,

$$\int \frac{dx}{\cot x} = -l(\cos x) \dots \dots (4)$$

5. Let $dy = \frac{dx}{\sin x \cos x}$.

From trigonometry, we have,

$$\sin x \cos x = \frac{1}{2} \sin 2x;$$

we also have,

$$dx = \frac{1}{2} d(2x);$$

hence,

$$\frac{dx}{\sin x \cos x} = \frac{d(2x)}{\sin 2x}.$$

Applying Formula (1), we have,

$$\int \frac{dx}{\sin x \cos x} = l(\tan x) \dots \dots (5)$$

Let us have the expression,

$$dy = \sin^m x \cos^n x dx \dots \dots (6)$$

Making $\sin x = z$, whence $\cos x = \sqrt{1 - z^2}$,

$$\text{and } dx = \frac{dz}{\sqrt{1 - z^2}},$$

we have,

$$dy = z^m (1 - z^2)^{\frac{n-1}{2}} dz \dots \dots (7)$$

Form (7) can always be integrated when m and n are whole numbers, because it will then satisfy one of the *three* criterions (Art. 69).

6. $dy = \sin^2 x \cos^2 x dx$.

Comparing with (6), we find $m = 2$, and $n = 3$; hence, from (7), we have,

$$dy = (1 - z^2)z^2 dz; \quad \therefore y = \frac{z^3}{3} - \frac{z^5}{5} = \sin^3 x \left(\frac{1}{3} - \frac{1}{5} \sin^2 x \right).$$

7. Let $dy = \sin^3 x dx$.

Here $m = 3$, and $n = 0$; and $dy = (1 - z^2)^{-\frac{1}{2}} z^2 dz$; making $x = z$, $a = 1$, $b = -1$, $n = 2$, $p = -\frac{1}{2}$, and $m = 4$, in Formula A, we have,

$$\int (1 - z^2)^{-\frac{1}{2}} z^2 dz = -\frac{(1 - z^2)^{\frac{1}{2}} z^3}{3} + \frac{2}{3} \int (1 - z^2)^{-\frac{1}{2}} z dz.$$

But from Formula (29),

$$\int (1 - z^2)^{-\frac{1}{2}} z dz = -(1 - z^2)^{\frac{1}{2}};$$

hence,

$$y = -\frac{1}{3} (1 - z^2)^{\frac{1}{2}} z^3 - \frac{2}{3} (1 - z^2)^{\frac{1}{2}} = -\frac{1}{3} \cos x (\sin^3 x + 2).$$

8. Let $dy = \cos^2 x dx = dz \sqrt{1 - z^2}$.

Applying Formula (B), we find,

$$\int (1 - z^2)^{\frac{1}{2}} dz = \frac{1}{2} (1 - z^2)^{\frac{1}{2}} z + \frac{1}{2} \int \frac{dz}{(1 - z^2)^{\frac{1}{2}}}.$$

But,

$$\int \frac{dz}{(1 - z^2)^{\frac{1}{2}}} = \int dx = x.$$

Hence,

$$y = \frac{1}{2}(\cos x \sin x + x).$$

$$9. \text{ Let } dy = \frac{dx}{\sin^3 x} = \frac{dz}{z^3 \sqrt{1-z^2}} = (1-z^2)^{-\frac{1}{2}} z^{-3} dz.$$

Applying Formula C, we have,

$$\begin{aligned} \int (1-z^2)^{-\frac{1}{2}} z^{-3} dz &= -\frac{(1-z^2)^{\frac{1}{2}} z^{-2}}{2} \\ &\quad + \frac{1}{2} \int (1-z^2)^{-\frac{1}{2}} z^{-1} dz. \end{aligned}$$

But,

$$\int (1-z^2)^{-\frac{1}{2}} z^{-1} dz = \int \frac{dz}{z \sqrt{1-z^2}} = \int \frac{dx}{\sin x} = l(\tan \frac{1}{2} x).$$

Hence,

$$y = -\frac{\cos x}{2 \sin^2 x} + \frac{1}{2} l(\tan \frac{1}{2} x).$$

$$10. \text{ Let } dy = \frac{dx}{\sin x \cos^3 x} = \frac{dz}{z(1-z^2)^2} = (1-z^2)^{-2} z^{-1} dz.$$

By Formula D, we have,

$$\int (1-z^2)^{-2} z^{-1} dz = \frac{1}{2} (1-z^2)^{-1} + \int (1-z^2)^{-1} z^{-1} dz.$$

But,

$$\begin{aligned} \int (1-z^2)^{-1} z^{-1} dz &= \int \frac{dz}{z \sqrt{1-z^2} \sqrt{1-z^2}} \\ &= \int \frac{dx}{\sin x \cos x} = l(\tan x). \end{aligned}$$

Hence,

$$y = \frac{1}{2 \cos^2 x} + l(\tan x).$$

11. Let $dy = \tan^4 x dx = (1 - z^2)^{-\frac{1}{2}} z^4 dz$,

$$y = \frac{1}{3} \tan^3 x - \tan x + x.$$

EXAMPLES IN SUCCESSIVE INTEGRATION.

12. $d^2y = \sin x \cos^2 x dx^2 = \sin x (d \sin x)^2$.

Making $\sin x = z$, whence, $d^2y = z(dz)^2$, we have,

$$\frac{dy}{dz} = \frac{z^3}{2} + C, \text{ and } y = \frac{z^3}{6} + Cz + C',$$

or,

$$y = \frac{\sin^3 x}{6} + C \sin x + C'.$$

13. $d^2y = \cos x \sin^2 x dx^2 = \cos x (d \cos x)^2$;

$$\therefore \frac{dy}{d(\cos x)} = \frac{1}{2} \cos^2 x + C,$$

and

$$y = \frac{1}{6} \cos^3 x + C \cos x + C'.$$

14. $d^2y = -\frac{dx^2}{x^3}$;

$$\therefore \frac{dy}{dx} = \frac{1}{x} + C; \text{ or, } y = \ln x + Cx + C'$$

Integration of Differential Functions of two Variables.

77. A total differential of a function of two variables is of the form,

$$dz = Pdx + Qdy \dots \dots (1)$$

In which P and Q are functions of x and y ; but differential expressions of that form are not necessarily total differentials. That they may be so, they must (Eq. 5, Art. 26), satisfy the condition,

$$\frac{dP}{dy} = \frac{dQ}{dx} \dots \dots (2)$$

When an expression of the form (1) is to be integrated, we first apply the test expressed by (2); if that be satisfied, the expression is *integrable*. To perform the integration, integrate one of the partial differentials with reference to its corresponding variable, that is, as though the other variable were constant, and add such a function of that variable as will satisfy equation (1). Thus, to integrate expression (1), we have,

$$z = \int P dx + R \dots \dots (3)$$

in which R is a function of y alone. The value of R may be found by differentiating the second member of (3) with respect to y , and placing its partial differential coefficient equal to Q . Hence, we have,

$$\frac{d \int P dx}{dy} + \frac{dR}{dy} = Q.$$

Transposing, multiplying by dy , and integrating with respect to y , that is, as though x were constant, we find,

$$R = \int \left(Q - \frac{d \int P dx}{dy} \right) dy.$$

Substituting in (3), we find,

$$z = \int P dx + \int \left(Q - \frac{d \int P dx}{dy} \right) dy \dots \dots (4)$$

EXAMPLES.

1. $dz = 3x^2y^2 dx + 2x^3y dy.$

Here $P = 3x^2y^2$, and $Q = 2x^3y$, and (2) is satisfied. Integrating by Formula (4), we have,

$$z = x^3y^2 + \int (2x^3y - 2x^3y) dy = x^3y^2 + C.$$

2. $dz = y dx + x dy.$

The test (2) is satisfied. Integrating by (4), we have,

$$z = yx + \int (x dy - x dy) dy = yx + C.$$

3. $dz = \frac{dx}{y} + \left(2y - \frac{x}{y^2} \right) dy.$

By Formula (4), we have,

$$z = \frac{x}{y} + \int \left(2y - \frac{x}{y^2} + \frac{x}{y^2} \right) dy = \frac{x}{y} + y^2 + C.$$

4. $dz = (6xy - y^2) dx + (3x^2 - 2xy) dy.$

$$\begin{aligned} z &= 3x^2y - y^2x + \int (3x^2 - 2xy - 3x^2 + 2xy) dy \\ &= 3x^2y - y^2x + C. \end{aligned}$$

PART IV.

APPLICATIONS OF THE INTEGRAL CALCULUS.

I. LENGTHS OF PLANE CURVES.

Rectification.

78. *Rectification* is the operation of finding an expression for the length of a curve. This may be done by finding an element of the length, as explained in Art. 51, and then integrating it between proper limits. If we apply the integral sign to Equation (1), Art. 51, we have,

$$L = \int \sqrt{dx^2 + dy^2} \dots \dots (1)$$

1°. *To rectify the semi-cubic parabola, whose equation is,*

$$y^3 = a^2x^2.$$

Differentiating and substituting in (1), we have,

$$L = \frac{1}{2a} \int (4a^3 + 9y)^{\frac{1}{2}} dy \dots \dots (2)$$

Integrating by Formula (29), we have,

$$L = \frac{1}{27a} (4a^3 + 9y)^{\frac{3}{2}} + C \dots \dots (3)$$

As explained in Art. 60, this integral is indefinite, and

expresses the length of an arc from any ordinate up to any other ordinate.

If we estimate the length from a point whose ordinate is 0, the length at that point is 0, and, from (3), making L , and $y = 0$, we have,

$$0 = \frac{1}{27a} (4a^3)^{\frac{3}{2}} + C; \therefore C = -\frac{8a^3}{27}.$$

Substituting this value of C in (3), and denoting the corresponding value of L by L' , we have,

$$L' = \frac{1}{27a} (4a^3 + 9y)^{\frac{3}{2}} - \frac{8a^3}{27} \dots\dots (4)$$

This is a *particular* integral, and it expresses the length of the arc from the particular ordinate 0, up to any ordinate y .

If we wish to find the length from the particular ordinate 0, up to an ordinate b , we make $y = b$, in (4). Doing so, and denoting the corresponding value of L' by L'' , we have,

$$L'' = \frac{1}{27a} (4a^3 + 9b)^{\frac{3}{2}} - \frac{8a^3}{27} \dots\dots (5)$$

This is the *definite* integral, and expresses the length of the arc from the ordinate 0, up to the ordinate b .

In this case the *limits* are 0 and b ; the integral may be found otherwise, as follows:

Making $y = 0$, in (3), and denote the corresponding value of L by L_0 , we have,

$$L_0 = \frac{1}{27a} (4a^3)^{\frac{3}{2}} + C \dots\dots (6)$$

Making $y = b$, in (3), and denoting the corresponding value of L by L_b , we have,

$$L_b = \frac{1}{27a} (4a^3 + 9b)^{\frac{3}{2}} + C \dots (7)$$

Equation (6) expresses the length of the curve from any point up to the point whose ordinate is 0, and (7) expresses the length of the arc from the same point up to a point whose ordinate is b ; the first taken from the second will therefore be the *definite* integral, and will express, as before, the length of the arc from the point whose ordinate is 0, up to the point whose ordinate is b . Making the subtraction, and adopting the notation explained in Art. 60, we have,

$$L'' = \frac{1}{2a} \int_0^b (4a^3 + 9y)^{\frac{1}{2}} dy = \frac{1}{27a} (4a^3 + 9b)^{\frac{3}{2}} - \frac{8a^3}{27} \dots (8)$$

The same result as found in (5). In this case the *initial* abscissa is 0 and the *terminal* abscissa is $\frac{b^{\frac{2}{3}}}{a}$.

2°. To rectify the cycloid, whose differential equation (Art. 56), is

$$dx = \pm \frac{y dy}{\sqrt{2ry - y^3}} \dots (9)$$

Substituting, in (1), and reducing, we have,

$$L = \sqrt{2r} \int (2r - y)^{-\frac{1}{2}} dy \dots (10)$$

Integrating between the limits 0 and $2r$ Formula (29), we have,

$$L'' = \sqrt{2r} \int_0^{2r} (2r - y)^{-\frac{1}{2}} dy = 4r \dots (11)$$

This is the expression for *half* of one branch. Hence, the whole branch is equal to eight times the radius of the generating circle.

3°. To rectify the circle, whose equation is,

$$y = (r^2 - x^2)^{\frac{1}{2}}.$$

Differentiating, substituting in (1), and developing by the binomial formula, we have,

$$L = r \int (r^2 - x^2)^{-\frac{1}{2}} dx = \int \left(dx + \frac{1}{2r^2} \cdot x^2 dx + \frac{1.3}{2.4r^4} \cdot x^4 dx + \frac{1.3.5}{2.4.6r^6} x^6 dx + \text{etc.} \right) \dots \dots (12)$$

Performing the integration, making $r = 1$, and commencing the arc at the point whose abscissa is 0, that is, at the upper extremity of the vertical diameter, we have,

$$L' = x + \frac{1}{3.2} x^3 + \frac{1.3}{5.2.4} x^5 + \frac{1.3.5}{7.2.4.6} x^7 + \text{etc.} \dots \dots (13)$$

Formula (13) may be used for finding the length of an arc, but the series is not very converging, and therefore the process is tedious. We know, when $x = \frac{1}{2}$, that L' is equal to an arc of 30° , that is, to $\frac{1}{6}\pi$. Making $L' = \frac{1}{6}\pi$, and $x = \frac{1}{2}$, we have,

$$\frac{1}{6}\pi = \frac{1}{2} + \frac{1}{2^3.2.3} + \frac{1.3}{2^5.2.4.5} + \frac{1.3.5}{2^7.2.4.6.7} + \text{etc.}, = .5235987; \\ \therefore \pi = 3.1415 \dots \dots (14)$$

4°. A better method for deducing the value of π , is to find the length of the arc, in terms of its tangent.

If $L = \tan^{-1}x$, we have (Art. 18),

$$dL = \frac{dx}{1+x^2} = (1+x^2)^{-1} dx.$$

Developing by the binomial formula, we have,

$$dL = dx(1 - x^2 + x^4 - x^6 + x^8 - \text{etc.}) \dots \dots (15)$$

Integrating, and determining C , as before, we have,

$$L' = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \text{etc.} \dots \dots (16)$$

To render (16) converging, x must be made small. Assume the trigonometrical formula,

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b},$$

and let $a = \tan^{-1}m$, $b = \tan^{-1}n$, and $a + b = \tan^{-1}z$; then will

$$z = \frac{m + n}{1 - mn}; \text{ or, } n = \frac{z - m}{mz + 1} \dots \dots (17)$$

If $z = 1$, and $m = \frac{1}{5}$, we have, $n = \frac{2}{3}$;

$$\therefore \tan^{-1} 1 = \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{2}{3}.$$

If $z = \frac{2}{3}$, and $m = \frac{1}{5}$, we have, $n = \frac{7}{17}$;

$$\therefore \tan^{-1} \frac{2}{3} = \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{7}{17}.$$

If $z = \frac{7}{17}$, and $m = \frac{1}{5}$, we have, $n = \frac{9}{46}$;

$$\therefore \tan^{-1} \frac{7}{17} = \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{9}{46}.$$

If $z = \frac{9}{46}$, and $m = \frac{1}{5}$, we have, $n = -\frac{1}{239}$;

$$\therefore \tan^{-1} \frac{9}{46} = \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}.$$

Hence, by successive substitution, we find,

$$\frac{\pi}{4} = \tan^{-1} 1 = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \dots \dots (18)$$

If we make $x = \frac{1}{5}$, in Equation (16), we find the value of $\tan^{-1} \frac{1}{5}$, and in like manner, we find the $\tan^{-1} \frac{1}{239}$; substituting them in (18), and reducing, we have the value of π . With very little labor, the value of π may be found to 8 or 10 places of decimals.

II. AREAS OF PLANE CURVES.

Quadrature.

79. *Quadrature* is the operation of finding an expression for the area of a portion of a plane bounded by a curve, the axis of abscissas, and any two ordinates. This is done by finding an expression for the elementary area, as explained in Art. 51, and then integrating the result between proper limits. Applying the sign of integration to Formula (2), Art. 51, we have,

$$A = \int y dx \dots \dots (1)$$

1°. *To find the area of a parabola, whose equation is,*

$$y^2 = 2px.$$

Finding the value of y , and substituting in (1), we have,

$$A = \sqrt{2p} \int x^{\frac{1}{2}} dx = \frac{2}{3} \sqrt{2p} \cdot x^{\frac{3}{2}} + C \dots \dots (2)$$

If we commence the area at the ordinate 0, we have, $C = 0$; hence,

$$A' = \frac{2}{3} \sqrt{2p} \cdot x^{\frac{3}{2}} = \frac{2}{3} yx \dots \dots (3)$$

That is, the area of any portion of a parabola, reckoned from the vertex, is two-thirds the rectangle of its terminal co-ordinates.

2°. To find the area of a circle, whose equation is,

$$y = \sqrt{r^2 - x^2}.$$

Substituting in (1), we find,

$$A = \int (r^2 - x^2)^{\frac{1}{2}} dx \dots \dots (4)$$

Reducing by Formula B, and integrating the last term by Formula (20), we have,

$$A = \frac{x(r^2 - x^2)^{\frac{1}{2}}}{2} + \frac{r^2}{2} \sin^{-1} \frac{x}{r} + C \dots \dots (5)$$

Taking the integral between the limits, $x = 0$ and $x = r$, which gives the area of a quadrant, and remembering that $\sin^{-1} 1 - \sin^{-1} 0 = \frac{1}{2}\pi$, we have,

$$A'' = \frac{r^2}{2} [\sin^{-1} 1 - \sin^{-1} 0] = \frac{\pi r^2}{4}; \therefore \text{Area} = \pi r^2 \dots (6)$$

3°. To find the area of the ellipse, whose equation is,

$$y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}.$$

Substituting in (1), and integrating, as in the last example, we have,

$$A = \frac{b}{a} \left[\frac{x(a^2 - x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] + C \dots \dots (7)$$

Integrating from $x = 0$ to $x = a$, and multiplying by 4, we have, for the area of the ellipse,

$$4A'' = \pi ab \dots \dots (8)$$

4°. To find the area of the cycloid, whose differential equation is,

$$dx = \pm \frac{ydy}{\sqrt{2ry - y^2}}$$

Substituting in (1), we find,

$$A = \int \frac{y^2 dy}{\sqrt{2ry - y^2}} \dots \dots (9)$$

Applying Formula *E* twice, and Formula (26) once, we find,

$$A = - \frac{y\sqrt{2ry - y^2}}{2} + \frac{3r}{2} \left[-\sqrt{2ry - y^2} + r \text{versin}^{-1} \frac{y}{r} \right] + C.$$

Taking the integral between the limits $y = 0$, and $y = 2r$, and multiplying by 2, we find, for the area of the cycloid,

$$2A'' = 3\pi r^2 \dots \dots (10)$$

That is, the area between one branch and the directing line is three times the area of the generating circle.

5°. To find the area of the logarithmic curve, whose equation is, $y = lx$.

Substituting in (1), we have,

$$A = \int (lx) dx \dots (11)$$

Making $m = 1$ and $n = 1$, in Formula F , and reducing, we have,

$$A = xlx - x + C \dots (12)$$

If we commence the arc at the ordinate corresponding to $x = 1$, we have, $C = 1$, and

$$A' = x(lx - 1) + 1 \dots (13)$$

6°. *To find the area of a rectangular hyperbola, bounded by the curve, one asymptote, and any two ordinates to that asymptote.*

Assume the equation,

$$xy = m, \text{ and make } m = 1, \text{ whence } y = \frac{1}{x}.$$

Substituting in (1), we have,

$$A = \int \frac{dx}{x} = lx + C \dots (14)$$

Commencing the area from the ordinate through the vertex, where $x = 1$, we have, $C = 0$; hence,

$$A' = lx \dots (15)$$

That is, the area commencing from the ordinate through the vertex is the Napierian logarithm of the terminal abscissa. Had we not made $m = 1$, we should, in like manner, have found

$$A' = mlx.$$

In which the area is equal to the logarithm in a system whose modulus is m .

III. AREAS OF SURFACES OF REVOLUTION.

Surfaces Generated by the Revolution of Plane Curves.

80. The area of a portion of a surface of revolution, bounded by two planes perpendicular to its axis, is determined by finding an expression for an elementary zone, as explained in Art. 51, and integrating the result between proper limits. Applying the sign of integration to Formula (3), Art. 51, we have,

$$S = \int 2\pi y \sqrt{dx^2 + dy^2} \dots\dots (1)$$

1°. *To find the surface of a sphere, the equation of the generating circle being, $y = (r^2 - x^2)^{\frac{1}{2}}$.*

Differentiating, we have, $dy = -(r^2 - x^2)^{-\frac{1}{2}} x dx$; substituting the values of y and dy in (1), and integrating from $-r$ to $+r$, we have,

$$S'' = 2\pi r \int_{-r}^r dx = 4\pi r^2 \dots\dots (2)$$

Hence, the area of the surface of a sphere is equal to four great circles, or to two-thirds the surface of the circumscribed cylinder.

2°. *To find the surface of a right cone.*

The equation of the generating line, the vertex of the cone being at the origin, is

$$y = ax; \therefore dy = a dx.$$

Substituting in (1), and reducing, we have,

$$S = 2\pi a\sqrt{1+a^2} \int x dx = \pi a x^2 \sqrt{1+a^2} + C \dots \dots (3)$$

If the initial plane pass through the vertex, we have, for that plane, $S = 0$, and $x = 0$, whence $C = 0$; hence,

$$S' = \pi a x^2 \sqrt{1+a^2} = \pi y \times x \sqrt{1+a^2} \dots \dots (4)$$

But a is the tangent of the semi-angle of the cone, and consequently $x\sqrt{1+a^2}$ is the slant height; $2\pi y$ is the circumference of the cone's base; hence, the convex surface is equal to half the circumference of the base into the slant height.

3°. *To find the surface of the paraboloid of revolution.*

Assume the equation $y^2 = 2px$, whence,

$$y = \sqrt{2px}, \text{ and } dy^2 = \frac{p}{2x} dx^2.$$

Substituting in (1), and reducing, we have,

$$S = 2\pi \int (p^2 + 2px)^{\frac{1}{2}} dx = \frac{2\pi}{3p} (p^2 + 2px)^{\frac{3}{2}} + C \dots \dots (5)$$

If the initial plane pass through the vertex, we have,

$$0 = \frac{2\pi p^2}{3} + C; \therefore C = -\frac{2\pi p^2}{3}.$$

Which, in (5), gives,

$$S' = \frac{2\pi}{3} \left[\frac{1}{p} (p^2 + 2px)^{\frac{3}{2}} - p^2 \right] \dots \dots (6)$$

4°. *To find the surface generated by revolving one branch of a cycloid about its base.*

Assuming the equation, $dx = \pm \frac{ydy}{\sqrt{2ry - y^2}}$, and substituting in (1), we have,

$$S = 2\pi\sqrt{2r} \int (2r - y)^{-\frac{1}{2}} y dy \dots\dots (7)$$

Applying Formula A, and integrating the last term by Formula (29), we have,

$$S = -2\pi\sqrt{2r} \left[\frac{2}{3} y(2r - y)^{\frac{1}{2}} + \frac{8r}{3} (2r - y)^{\frac{1}{2}} \right] + C.$$

Integrating between the limits $y = 0$ and $y = 2r$, we have,

$$S'' = \frac{32}{3} \pi r^2 \dots\dots (8)$$

This is the surface generated by half of one branch. Hence, to find the whole surface, we multiply by 2. This gives

$$2S'' = \frac{64}{3} \pi r^2.$$

IV. VOLUMES OF SOLIDS OF REVOLUTION.

Cubature.

81. *Cubature* is the operation of finding the volume of a solid. When this is bounded by a surface of revolution and two planes perpendicular to its axis, the volume may be ascertained by finding the volume of an element bounded by two such planes infinitely near to each other, as explained in Art. 51, and then integrating the

result between proper limits. Applying the sign of integration to Formula (4), Art. 51, we have,

$$V = \pi \int y^2 dx \dots\dots (1)$$

1°. *To find the volume of a sphere.*

Assume the equation, $y^2 = r^2 - x^2$, and combine it with (1); we have,

$$V = \pi \int (r^2 dx - x^2 dx) = \pi \left[r^2 x - \frac{x^3}{3} \right] + C.$$

Taking the integral between the limits, $x = -r$, and $x = +r$, we find,

$$V'' = \pi \left[2r^3 - \frac{2r^3}{3} \right] = \frac{4}{3} \pi r^3 = 4\pi r^2 \times \frac{1}{3} r \dots\dots (2)$$

That is, *the volume is equal to the surface by one-third the radius.*

2°. *To find the volume of a spheroid of revolution.*

There are two species of spheroids of revolution.

1st. The *prolate spheroid*, generated by revolving an ellipse about its transverse axis.

2dly. The *oblate spheroid*, generated by revolving an ellipse about its conjugate axis.

1st. *The prolate spheroid.* In this case the equation of the meridian curve is, $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$. Substituting this in (1), and integrating between the limits $x = -a$, and $x = +a$, we have,

$$V'' = \pi \frac{b^2}{a^2} \int_{-a}^{+a} (a^2 - x^2) dx = \frac{4}{3} \pi b^2 a = \frac{2}{3} \pi b^2 \times 2a \dots\dots (3)$$

That is, *the volume is equal to two-thirds of the circumscribing cylinder.*

2dly. *The oblate spheroid.* In this case, if the conjugate axis coincide with the axis of x , the equation of the meridian curve is, $y^2 = \frac{a^2}{b^2}(b^2 - x^2)$. Substituting in (1), and integrating from $-b$ to $+b$, we have,

$$V'' = \frac{4}{3}\pi a^2 b = \frac{2}{3}\pi a^2 \times 2b \dots\dots (4)$$

Hence, as before, we have, *the volume equal to two-thirds the circumscribing cylinder.*

In both cases, if $a = b = r$, we have,

$$V'' = \frac{4}{3}\pi r^3 \dots\dots (5)$$

This hypothesis causes the ellipsoids to merge into the sphere.

3°. *To find the volume of a paraboloid of revolution.*

The equation of the meridian curve is, $y^2 = 2px$. Hence, from (1), we have,

$$V = 2\pi p \int x dx = \pi p x^2 + C \dots\dots (6)$$

If the initial plane pass through the vertex, we have $C = 0$, and

$$V' = \pi p x^2 = \pi y^2 \times \frac{1}{2}x \dots\dots (7)$$

That is, *the volume is equal to half the cylinder that has the same base and the same altitude.*

4°. To find the volume generated by revolving one branch of the cycloid about its base.

The differential equation of the meridian curve is,

$$dx = \frac{ydy}{\sqrt{2ry - y^2}};$$

hence, from (1), we have,

$$V = \pi \int \frac{y^3 dy}{\sqrt{2ry - y^2}}.$$

Reducing by Formula E, integrating the last term by Formula (26), and taking the integral between the limits, $y = 0$, and $y = 2r$, we find, for one-half the volume required,

$$V'' = \frac{5}{2} \pi^2 r^3, \text{ or } 2V'' = 5\pi^2 r^3 = \frac{5}{8} \pi (2r)^3 \times 2\pi r \dots (8)$$

Hence, the volume is equal to five-eighths the circumscribing cylinder.

5°. To find the volume generated by revolving the logarithmic curve about the axis of numbers.

The equation of the generatrix is $y = lx$. Hence,

$$V = \pi \int (lx)^2 dx \dots (9)$$

Reducing by Formula F, we have,

$$V = \pi [x(lx)^2 - 2(xlx - x)] + C.$$

If the initial plane pass through the point whose abscissa is 1, we have, $C = -2\pi$. Hence,

$$V' = \pi [x(lx)^2 - 2(xlx - x + 1)] \dots (10)$$

PART V.

APPLICATIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS TO MECHANICS AND ASTRONOMY.

I. CENTRE OF GRAVITY.

82. In what follows, bodies are supposed to be homogeneous; the weight of any part of a body is therefore proportional to its volume, and consequently the weight of the unit of volume may be taken as the unit of weight. Points, lines, and surfaces are supposed to be material: A *material point* is a body whose length, breadth, and thickness are infinitesimal; a *material line* is a line whose length is finite, and whose breadth and thickness are infinitesimal; a *material surface* is a body whose length and breadth are finite, and whose thickness is infinitesimal; under this supposition a point is an elementary portion of a line, a line is an elementary portion of a surface, and a surface is an elementary portion of a solid.

The weights of the elements of a body are directed toward the centre of the earth, and because the bodies treated of are exceedingly small in comparison with the earth, these weights may be regarded as a system of parallel forces; hence, the weight of a body is equal to the sum of the weights of its elements and is parallel to them.

The *centre of gravity* of a body is a point through which its weight always passes. This point may be found by the *principle of moments*, which may be enunciated as follows: *the moment of the resultant of any number of forces, with respect to an axis, is equal to the algebraic sum of the moments of the forces, with respect to the same axis.* (*Mechanics*, Art. 35.)

In applying this principle, we assume the following results of demonstrations in mechanics: 1°. The centre of gravity of a straight line is at its middle point; 2°. The centre of gravity of a plane figure is in that plane; 3°. If a plane figure have a line of symmetry, its centre of gravity is on that line; and 4°. If a solid have a plane of symmetry, its centre of gravity is in that plane. (*Mechanics*, Arts. 44, 45, 46.)

To deduce general formulas for finding the centre of gravity of a body, assume a system of co-ordinate axes that are to retain a fixed position with respect to the body, but that change position when the body moves; denote the volume, and consequently the weight, of any element of the body by dv , and the co-ordinates of its centre of gravity by x , y , and z ; denote the weight of the body by

$v = \int dv$, and the co-ordinates of its centre of gravity by x_1 , y_1 , and z_1 .

If the body be placed in such a position that the plane xy is horizontal, the weights of the elements and of the body are parallel to the axis of z , the moment of the body,

with respect to the axis of y , is $x_1 \int dv$, the moment of any element, with respect to the same axis, is $x dv$,

and the algebraic sum of the moments of all the elements is $\int xdv$; hence, from the principle of moments, we have,

$$x_1 \int dv = \int xdv; \therefore x_1 = \frac{\int xdv}{\int dv} \dots\dots (1)$$

In like manner, we have,

$$y_1 \int dv = \int ydv; \therefore y_1 = \frac{\int ydv}{\int dv} \dots\dots (2)$$

If the body be turned about so that the plane yz is horizontal, we have, in like manner,

$$z_1 \int dv = \int zdv; \therefore z_1 = \frac{\int zdv}{\int dv} \dots\dots (3)$$

When the body is in a plane, that plane may be taken as the plane xy , in which case $z_1 = 0$; if the body have an axis of symmetry, that may be taken as the axis of x , in which case $z_1 = 0$, and $y_1 = 0$.

Centre of Gravity of a Circular Arc.

83. Let the radius perpendicular to the chord of the arc be taken as the axis of x ; then will z_1 , and y_1 be equal to 0. Denote the radius of the circle by r , the chord by c , and the arc by A . The origin being at the centre, the equation of the arc is, $y^2 = r^2 - x^2$; hence,

$$dv = \sqrt{dx^2 + dy^2} = \sqrt{\frac{y^2}{x^2} dy^2 + dy^2}$$

$$= \frac{r dy}{x} = \frac{r dy}{\sqrt{r^2 - y^2}};$$

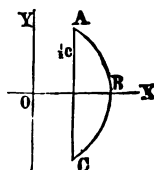


Fig. 16.

Substituting in (1), and integrating between the limits, $y = -\frac{1}{2}c$, and $y = +\frac{1}{2}c$, we have, for the numerator,

$$\int_{-\frac{1}{2}c}^{+\frac{1}{2}c} r dy = rc,$$

and for the denominator,

$$\int_{-\frac{1}{2}c}^{+\frac{1}{2}c} \frac{r dy}{\sqrt{r^2 - y^2}} = r \left[\sin^{-1} \frac{c}{2r} - \sin^{-1} \frac{-c}{2r} \right] = \text{arc } ABC.$$

Hence,

$$x_1 = \frac{rc}{\text{arc } ABC}, \text{ or } \text{arc } ABC : c :: r : x_1.$$

That is, *the centre of gravity of the arc of a circle is on the diameter that bisects the chord, and its distance from the centre is a fourth proportional to the arc, its chord, and the radius.*

Centre of Gravity of a Parabolic Area.

84. Let the area be limited by a double ordinate, and denote the extreme abscissa by a . From the equation of the curve, $y^2 = 2px$, we have,

$$y = \sqrt{2p} \cdot x^{\frac{1}{2}}.$$

$$\therefore dv = y dx = \sqrt{2p} \cdot x^{\frac{1}{2}} dx,$$

$$\text{and } x dv = \sqrt{2p} \cdot x^{\frac{3}{2}} dx.$$

Substituting in (1), and integrating between the limits 0 and a , we have,

$$x_1 = \frac{3}{5} a.$$

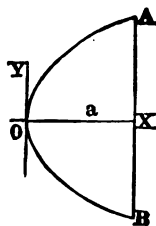


Fig. 17.

That is, *the centre of gravity is on the axis, at a distance from the vertex equal to three-fifths the altitude of the segment.*

Centre of Gravity of a Semi-ellipsoid of Revolution.

85. Let the axis of the ellipsoid be taken as the axis of x . Then, if the origin be taken at the centre, the equation of the generating curve is

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2).$$

In this case, we have,

$$dv = \pi y^2 dx = \pi \frac{b^2}{a^2} (a^2 - x^2) dx,$$

$$\text{and } xdv = \pi \frac{b^2}{a^2} (a^2 x - x^3) dx.$$

Substituting in (1), and integrating between the limits $x = 0$, and $x = a$, we have,

$$x_1 = \frac{3}{8} a = \frac{3}{16} 2a.$$

That is, *the centre of gravity of a semi-prolate spheroid of revolution is on its axis of revolution, and at a distance from the centre equal to three-sixteenths the major axis of the generating ellipse.*

If we change a to b , and b to a , we find for the semi-oblate spheroid,

$$x_1 = \frac{3}{8} b = \frac{3}{16} 2b.$$

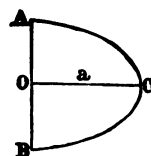


Fig. 18.

Centre of Gravity of a Cone.

86. Let the axis of the cone be the axis of x , and the vertex of the cone at the origin; denote the altitude by h , and the radius of the base by r ; then will the tangent of the semi-angle of the cone be $\frac{r}{h}$, the equation of the generating line will be $y = \frac{r}{h}x$, and we have,

$$dv = \pi y^2 dx = \pi \frac{r^2}{h^2} x^2 dx, \text{ and } xdv = \pi \frac{r^2}{h^2} x^3 dx.$$

Substituting in (1), and integrating between the limits 0 and h , we have,

$$x_1 = \frac{3}{4}h.$$

Centre of Gravity of a Paraboloid of Revolution.

87. Let the axis of the paraboloid be taken as the axis of x . The equation of the parabola being $y^2 = 2px$, we have,

$$dv = 2\pi p x dx, \text{ and } xdv = 2\pi p x^2 dx.$$

Substituting in (1), and integrating from $x = 0$ to $x = a$, we have,

$$x_1 = \frac{2}{3}a.$$

II. MOMENT OF INERTIA.**Definitions and Preliminary Principles.**

88. The *moment of inertia* of a body with respect to an axis, is equal to the algebraic sum of the products obtained by multiplying the mass of each element of the body by the square of its distance from the axis. If we take the axis through the centre of gravity of the body, and denote

the mass of an element by dm , its distance from the axis by x , and the moment of inertia by K , we have,

$$K = \int x^2 dm \dots \dots (1)$$

If we take any parallel axis at a distance d from the assumed axis, and denote the moment of inertia with respect to it by K' , we have (*Mechanics*, Art. 123).

$$K' = K + md^2 \dots \dots (2)$$

Moment of Inertia of a Straight Line.

89. Let the axis be taken through the centre of gravity of the line and perpendicular to it. Let AB represent the line, CD the axis, and E any element. Denote the length of the line by $2l$, its mass by m , the distance GE by x , and the length of the element by dx . From the principle of homogeneity, we have,

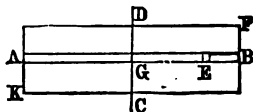


Fig. 19.

$$2l : dx :: m : dm ; \therefore dm = \frac{m}{2l} dx.$$

Substituting in (1), and integrating from $x = -l$ to $x = +l$, we have,

$$K = m \frac{l^2}{3} ; \therefore K' = m \left(\frac{l^2}{3} + d^2 \right) \dots \dots (3)$$

These formulas are entirely independent of the breadth of AB in the direction of the axis CD ; they hold good, therefore, when the filament is replaced by the rectangle KF , the axis being parallel to one of its ends. In this case m is the mass of the rectangle; $2l$, its length; and d , the distance of the axis from the centre of gravity of the rectangle.

Moment of Inertia of a Circle.

90. *First*, let the axis be taken to coincide with one of the diameters. Let ACB represent the circle, AB the axis, and $C'D'$ an element parallel to the axis. Denote OC by r , OE by x , EF by dx , $C'D'$ by $2y$, and the mass of the circle by m . Then, because the circle is homogeneous, we have,

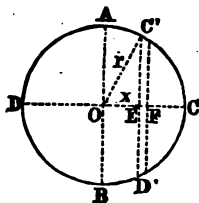


Fig. 20.

$$\pi r^2 : 2y dx :: m : dm ;$$

$$\therefore dm = \frac{2my}{\pi r^2} dx = \frac{2m\sqrt{r^2 - x^2}}{\pi r^2} dx.$$

Substituting in (1),

$$K = \frac{2m}{\pi r^2} \int (r^2 - x^2)^{\frac{1}{2}} x^2 dx.$$

Reducing by Formulas *A* and *B*, integrating by (20), and taking the integral between the limits $x = -r$ and $x = +r$, we find,

$$K = \frac{mr^2}{4} ; \quad \therefore K' = m\left(\frac{r^2}{4} + d^2\right) \dots \dots (4)$$

Secondly, let the axis be taken through the centre and perpendicular to the plane of the circle. Let KL be an elementary ring, whose radius is x , and whose breadth is dx ; then will its area be $2\pi x dx$, and from the principle, that the masses are proportional to the volumes, we have,

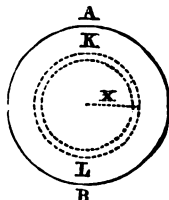


Fig. 21.

$$\pi r^2 : 2\pi x dx :: m : dm ; \quad \therefore dm = \frac{2mx dx}{r^2}$$

Substituting in (1), and integrating from $x = 0$ to $x = r$, we have,

$$K = \frac{mr^2}{2}; \therefore K' = m\left(\frac{r^2}{2} + d^2\right) \dots \dots (5)$$

Thirdly, to find the moment of inertia of a circular ring with respect to an axis perpendicular to its plane. Let m denote the mass of the ring, r and r' its extreme radii;

Then, $\pi(r^2 - r'^2) : 2\pi x dx :: m : dm$;

$$\therefore dm = \frac{2mx}{r^2 - r'^2} dx.$$

Substituting in (1) and integrating from $x = r'$ to $x = r$, we have

$$K = \frac{m(r^2 + r'^2)}{2}; \therefore K' = m\left(\frac{r^2 + r'^2}{2} + d^2\right) \dots (6)$$

Formulas (5) and (6) are independent of the thickness of the plate; hence, they will be true whatever that thickness may be. Hence, they hold good for a solid and hollow cylinder, m being taken to represent the mass of the cylinder.

Moment of Inertia of a Cylinder with respect to an Axis perpendicular to the Axis of the Cylinder.

91. Let the axis be taken through the centre of gravity of the cylinder, and let EF be an element perpendicular to the axis of the cylinder. Denote the length of the cylinder by $2l$, its radius by r , its mass by m , the distance of EF from the axis by x , and the thickness by dx . Then, as before, we have,

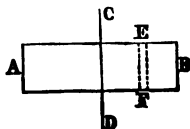


Fig. 22.

$$2l : dx :: m : dm; \therefore dm = \frac{m}{2l} dx.$$

The moment of inertia of this element is given by Formula (4); but this is the differential of the moment of inertia of the entire cylinder; hence, substituting dm for m , and x for d , in (4), we have,

$$dK = \frac{m}{2l} \left(\frac{r^2}{4} + x^2 \right) dx.$$

Integrating from $-l$ to $+l$, we have,

$$K = m \left(\frac{r^2}{4} + \frac{l^2}{3} \right); \quad \therefore K' = m \left(\frac{r^2}{4} + \frac{l^2}{3} + d^2 \right) \dots\dots (7)$$

Moment of Inertia of a Sphere.

92. Let the axis pass through the centre of a sphere. Let $C'D'$ be an elementary segment perpendicular to the axis, DC , whose volume is $\pi y^2 dx$ $= \pi(r^2 - x^2)dx$; its mass is found from the proportion,

$$\frac{4}{3}\pi r^3 : \pi(r^2 - x^2)dx :: m : dm;$$

$$\therefore dm = \frac{3m}{4r^3}(r^2 - x^2)dx.$$

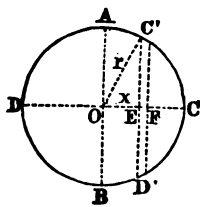


Fig. 24.

Substituting this value of dm for m in equation (5), and making r in that equation equal to y , or to $\sqrt{r^2 - x^2}$, we have, for the differential of the moment of inertia,

$$dK = \frac{3m}{8r^3}(r^2 - x^2)^2 dx.$$

Integrating between the limits $x = -r$, and $x = +r$, we have,

$$K = \frac{2mr^2}{5}; \quad \therefore K' = m \left(\frac{2}{5}r^2 + d^2 \right).$$

III. MOTION OF A MATERIAL POINT.

General Formulas.

93. *Uniform motion* is that in which the moving point passes over equal spaces in equal times; *variable motion* is that in which the moving point passes over unequal spaces in equal times. The *velocity* of a point is its *rate of motion*. The *acceleration* due to a force is the *rate of change* that it produces in the velocity of a point. When a force acts to increase the velocity, the acceleration is *positive*, when it acts to diminish the velocity, the acceleration is *negative*, and *conversely*.

Let us denote the *mass* of a material point by m , its *velocity* at any time t , by v , the *space* moved over at the time t , by s , and the *acceleration* due to the moving force by φ .

When the motion is uniform, *the velocity is constant*, and its measure is the space passed over in any time divided by that time; but we may, in all cases, regard the motion as uniform for an infinitely small time dt ; denoting the space described in that time by ds , we have,

$$v = \frac{ds}{dt} \dots \dots (1)$$

When the velocity varies uniformly *the acceleration is constant*, and we take for its measure the change of velocity in any time divided by that time; but we may, in all cases, regard the velocity as varying uniformly for an infinitely small time dt ; denoting the change of velocity in that time by dv , we have,

$$\varphi = \frac{dv}{dt} \dots \dots (2)$$

In the discussion of motion it is customary to regard the time as the independent variable. Differentiating (1), under that supposition, we find,

$$dv = \frac{d^2s}{dt^2};$$

substituting this in (2), we have,

$$\varphi = \frac{d^2s}{dt^2} \dots \dots (3)$$

Equations (1), (2), and (3) are fundamental, and from them we may deduce many laws of motion. If we multiply both members of (3) by m , we have,

$$m\varphi = m \frac{d^2s}{dt^2}; \text{ or, } F = m \frac{d^2s}{dt^2} \dots \dots (4)$$

In (4), F is the moving force. It will often be convenient to regard the mass of the material point as the unit of mass, in which case, we have, $m = 1$.

Uniformly Varied Motion.

94. *Uniformly varied motion* is that in which the velocity increases, or diminishes, uniformly. In the former case the motion is *uniformly accelerated*, in the latter it is *uniformly retarded*, in both the acceleration is *constant*. Denoting the constant value of φ by g , substituting in (3), multiplying both members by dt , and integrating, we have,

$$\frac{ds}{dt} = gt + C; \text{ or, } v = gt + C \dots \dots (5)$$

Multiplying again by dt , and integrating, we find,

$$s = \frac{1}{2}gt^2 + Ct + C' \dots \dots (6)$$

The constant, C , is what v becomes when $t = 0$; this is called the *initial velocity*, and may be denoted by v' . The constant, C' , is what s becomes when $t = 0$; this is called the *initial space*, and may be denoted by s' . Making these substitutions in (5) and (6), we have,

$$v = v' + gt \dots\dots (7)$$

$$s = s' + v't + \frac{1}{2}gt^2 \dots\dots (8)$$

Equations (7) and (8) enable us to discuss all the circumstances of uniformly varied motion (see *Mechanics*, Arts. 103 to 108). If g represent the force of gravity, and v' and s' be each equal to 0, (7) and (8) will express the laws of motion when a body falls from rest under the influence of gravity, regarded as a constant force. Under this supposition they become,

$$v = gt \dots\dots (9)$$

$$s = \frac{1}{2}gt^2 \dots\dots (10)$$

That is, the velocities generated are proportional to the times, and the spaces fallen through are proportional to the squares of the times.

Bodies Falling under the Influence of Gravity, regarded as Variable.

95. In accordance with the Newtonian law, the attraction exerted by the earth on a body at different distances varies inversely as the squares of their distances. Denoting the radius of the earth, supposed a sphere, by r , the force of gravity at the surface by g , any distance from the centre, greater than r , by s , and the force of gravity at that distance by ϕ , we have,

$$s^2 : r^2 :: g : \phi; \quad \therefore \phi = \frac{gr^2}{s^2}.$$

Substituting this value of ϕ in (3), and at the same time making it negative, because it acts in the direction of s negative, we have,

$$\frac{d^2s}{dt^2} = -\frac{gr^2}{s^3} \dots\dots (11)$$

Multiplying by $2ds$, and integrating both members, we have,

$$\frac{ds^2}{dt^2} = \frac{2gr^2}{s} + C; \text{ or, } v^2 = \frac{2gr^2}{s} + C \dots\dots (12)$$

If we make $v = 0$, when $s = h$, we have,

$$0 = \frac{2gr^2}{h} + C; \therefore C = -\frac{2gr^2}{h},$$

$$\text{and, } v^2 = 2gr^2 \left(\frac{1}{s} - \frac{1}{h} \right) \dots\dots (13)$$

Equation (13) gives the velocity generated whilst the body is falling from the height h to the height s . If we make $h = \infty$, and $s = r$, (13) becomes,

$$v^2 = 2gr; \therefore v = \sqrt{2gr} \dots\dots (14)$$

In this equation the resistance of the air is not considered.

If in (14) we make $g = 32.088$ feet, and $r = 20923596$ feet, their equatorial values, we find,

$$v = 36644 \text{ feet, or nearly } 7 \text{ miles per second.}$$

Equation (14) enables us to compute the velocity acquired by a body in falling from an infinite distance to the sun. If we make $g = 890.16$ feet, and $r = 430854.5$ miles, which are their values corresponding to the sun, we find,

$$v = 381 \text{ miles per second.}$$

If we make h equal to the distance of Neptune, and s equal to the sun's radius in (13), we find the velocity that a body would acquire in falling from Neptune to the sun, under the influence of the sun's attraction.

To find the time required for a body to fall through any space, substitute $\frac{ds}{dt}$ for v in (13), and solve the results with respect to dt ; this gives,

$$dt = -\sqrt{\frac{h}{2gr^2}} \cdot \frac{ds\sqrt{s}}{\sqrt{h-s}} = -\sqrt{\frac{h}{2gr^2}} \cdot \frac{sds}{\sqrt{hs-s^2}}. \quad (15)$$

The negative sign is taken because s decreases as t increases. Reducing (15) by Formula *E*, and integrating by Formula (26), we find,

$$t = \sqrt{\frac{h}{2gr^2}} \left[(hs - s^2)^{\frac{1}{2}} - \frac{h}{2} \text{versin}^{-1} \frac{2s}{h} \right] + C \dots \dots (16)$$

If $t = 0$ when $s = h$, we have,

$$C = \sqrt{\frac{h}{2gr^2}} \times \frac{\pi h}{2}.$$

This, in (16), gives,

$$t = \sqrt{\frac{h}{2gr^2}} \left[(hs - s^2)^{\frac{1}{2}} - \frac{1}{2} h \text{versin}^{-1} \frac{2s}{h} + \frac{1}{2} \pi h \right] \dots (17)$$

Making $s = r$ in (17), we find,

$$t = \sqrt{\frac{h}{2gr^2}} \left[(hr - r^2)^{\frac{1}{2}} - \frac{1}{2} h \text{versin}^{-1} \frac{2r}{h} + \frac{1}{2} \pi h \right] \dots (18)$$

Which gives the time required for a body to fall from a distance h , to the surface of the sun.

Bodies Falling under the Influence of a Force that varies as the Distance.

96. It will be shown hereafter, that if an opening were made along one of the diameters of the earth, and a body permitted to fall through it, the body would be urged toward the centre by a force varying as the distance from the centre, provided the earth were homogeneous. Assuming that principle, and denoting the force at the surface by g , the radius being r , and the force at the distance, s , from the centre by ϕ , we have,

$$r : s :: g : \phi; \therefore \phi = \frac{g}{r} s.$$

Substituting this value of ϕ in (3), and at the same time giving it the minus sign, because it acts in the direction of s negative, we have,

$$\frac{d^2 s}{dt^2} = -\frac{gs}{r}.$$

Multiplying by $2ds$, and integrating,

$$\frac{ds^2}{dt^2} = -\frac{gs^2}{r} + C; \text{ or, } v^2 = C - \frac{g}{r} s^2.$$

Making $v = 0$, when $s = r$, we have, $C = \frac{g}{r} r^2$; hence,

$$\frac{ds^2}{dt^2} = \frac{g}{r} (r^2 - s^2) \dots \dots (19)$$

If $s = 0$, we find $v = \sqrt{gr}$, which is the maximum velocity. It is equal to the entire velocity generated by a body falling from an infinite distance to the surface of the earth divided by $\sqrt{2}$. If the body pass the centre, s becomes negative, and we find the same values for v at equal dis-

tances from the centre, whether s be positive or negative. When s becomes equal to $-r$, v reduces to 0, and the body then falls toward the centre again, and so on continually.

Let us suppose $A'C'$ to be the diameter along which the body oscillates, and at the time the body starts from A' let a second body start from the same point and move around the semi-circumference, $A'MC'$, with a constant velocity equal to $\sqrt{gr}$. At any point, M , let this velocity be resolved into two components, MQ and MN , the former perpendicular and the latter parallel to $A'C'$. The latter component will be equal to

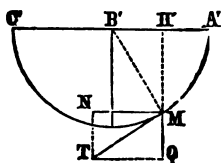


Fig. 25.

$\sqrt{gr}$, multiplied by $\cos TMN$, or its equal, $\cos B'MH'$; but $\cos B'MH' = \frac{H'M}{B'M}$; denoting $B'H'$ by s , whence $H'M$

$= \sqrt{r^2 - s^2}$, we have, $\cos B'MH'$

$= \frac{\sqrt{r^2 - s^2}}{r}$; hence, $MN = \sqrt{\frac{g}{r}} \cdot (r^2 - s^2)^{\frac{1}{2}}$; but this

is equal to the velocity of the oscillating point when at H' . Hence, the velocity of the vibrating point is everywhere equal to the parallel component of the velocity of the revolving point; they will, therefore, come together at the points A' and C' , and the position of the vibrating point will always be found by projecting the corresponding position of the revolving point on the path of the former. To find the time for a complete vibration from A' to C' , solve (19) with respect to dt , whence,

$$dt = - \sqrt{\frac{r}{g}} \cdot \frac{ds}{\sqrt{r^2 - s^2}} \dots \dots (20)$$

The negative sign is used because t is a decreasing function of s . Integrating (20) between the limits $s = r$ and $s = -r$, we find,

$$t'' = \pi \sqrt{\frac{r}{g}} \dots \dots (21)$$

This, we shall see hereafter, is the time of vibration of a simple pendulum whose length is r .

The time may be found otherwise, as follows: The space passed over by the revolving point is πr ; dividing this by the velocity, $\sqrt{gr}$, we have,

$$t = \pi \sqrt{\frac{r}{g}}$$

The species of vibratory motion just discussed is sometimes called harmonic, being the same as that of a point of a vibrating chord or spring.

Vibration of a Particle of an Elastic Medium.

97. It is assumed that if a particle of an elastic medium be slightly disturbed from its place of rest, and then abandoned, it will be urged back by a force that varies *directly* as the distance of the particle from its position of equilibrium; on reaching this position, the particle, by virtue of its inertia, will pass to the other side, again to be urged back, and so on.

Let us denote the displacement, at any time t , by s , and the acceleration due to the restoring force by ϕ ; then, from the law of force, we have, $\phi = n^2 s$, in which n is constant for the same medium, under the same circumstances of density, pressure, etc. Substituting for ϕ its value from Equation (3), and prefixing the negative sign, because it

acts in a direction contrary to that in which s is estimated, we have,

$$-\frac{d^2s}{dt^2} = n^2s \dots\dots (22)$$

Multiplying by $2ds$, and integrating, we have,

$$-\frac{ds^2}{dt^2} = n^2s^2 + C = -v^2 \dots\dots (23)$$

The velocity is 0 when the particle is at the greatest distance from the position of equilibrium; denoting this value of s by a , we have.

$$n^2a^2 + C = 0; \therefore C = -n^2a^2,$$

which, in (23), gives,

$$\frac{ds^2}{dt^2} = n^2(a^2 - s^2); \text{ or, } ndt = \frac{ds}{\sqrt{a^2 - s^2}} \dots\dots (24)$$

Integrating (24), we have.

$$nt + C = \sin^{-1} \frac{s}{a} \dots\dots (25)$$

Taking the sines of both members, and reducing, we have,

$$s = a \sin(nt + C) \dots\dots (26)$$

If we make $t = 0$, when $s = 0$, we have $C = 0$, and Equation (26) becomes,

$$s = a \sin(nt) \dots\dots (27)$$

If (25) be taken between the limits $-a$ and $+a$, we find, for the time of a single vibration, denoted by $\frac{1}{2}T$,

$$\frac{1}{2}T = \frac{\pi}{n}; \therefore n = \frac{2\pi}{T}.$$

Substituting this in (27), we have, finally,

$$s = a \sin\left(\frac{2\pi}{T} t\right) \dots\dots (28)$$

in which T is the time of a double vibration.

Solving (24), we have,

$$\frac{ds}{dt} = v = n\sqrt{a^2 - s^2} \dots\dots (29)$$

Substituting for s , in (29), its value from (28), we find,

$$v = n\sqrt{a^2 - a^2 \sin^2\left(\frac{2\pi}{T} t\right)} = na\sqrt{1 - \sin^2\left(\frac{2\pi}{T} t\right)}$$

Whence, we have,

$$v = na \cos\left(\frac{2\pi}{T} t\right) \dots\dots (30)$$

This equation is used in discussing the laws of light, and in many other cases.

Curvilinear Motion of a Point.

98. A point cannot move in a curve except under the action of an incessant force, whose direction is inclined to the direction of the motion. This force is called the *deflecting force*, and can be resolved into two components, one in the direction of the motion, and the other at right angles to it. The former acts simply to increase or diminish the velocity, and is called the *tangential force*: the latter acts to turn the point from its rectilinear direction, and being directed toward the centre of curvature is called the *centripetal force*. The resistance offered by the point to the centripetal force, in consequence of its

inertia, is equal and directly opposed to the centripetal force, and is called the *centrifugal force*.

To find expressions for the tangential and centripetal forces, let the acceleration due to the deflecting force in the direction of the axis of x , at any time t , be denoted by X , and the acceleration in the direction of the axis of y by Y . Let these be resolved into components acting tangentially and normally. The algebraic sum of the tangential components is the tangential acceleration denoted by T , and the algebraic sum of the normal components is the centripetal acceleration denoted by N . Assuming the notation of the figure, we have,

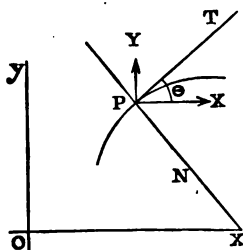


Fig. 26.

$$T = X \cos \theta + Y \sin \theta;$$

$$N = X \sin \theta - Y \cos \theta.$$

But from Articles (93) and (5), we have,

$$X = \frac{d^2 x}{dt^2}; \quad Y = \frac{d^2 y}{dt^2};$$

and,

$$\cos \theta = \frac{dx}{ds}; \quad \sin \theta = \frac{dy}{ds}.$$

Substituting in the preceding equations, and reducing on the supposition that t is the independent variable, and by means of Formula (3), Art. (38), we have,

$$\begin{aligned} T &= \frac{d^2 x}{dt^2} \cdot \frac{dx}{ds} + \frac{d^2 y}{dt^2} \cdot \frac{dy}{ds} \\ &= \frac{d(dx^2 + dy^2)}{dt^2 \cdot 2ds} = \frac{d(ds^2)}{dt^2 \cdot 2ds} = \frac{d^2 s}{dt^2} \dots \dots (31) \end{aligned}$$

$$\begin{aligned}
 N &= \frac{d^2 x dy - d^2 y dx}{dt^2 \cdot ds} \\
 &= -\frac{ds^2}{dt^2} \cdot \frac{d^2 y dx - d^2 x dy}{ds^2} = -\frac{v^2}{R}
 \end{aligned}$$

But the centripetal acceleration is equal and directly opposed to the centrifugal acceleration. Denoting the latter by f , we have,

$$f = \frac{v^2}{R} \dots \dots (32)$$

From (31) we see that the tangential force is independent of the centripetal force, and from (32) we see that the *acceleration due to the centrifugal force at any point of the trajectory, is equal to the square of the velocity divided by the radius of curvature at that point.*

Velocity of a Point rolling down a Curve in a Vertical Plane.

99. Let a point roll down a curve, situated in a vertical plane, under the influence of gravity regarded as constant. Let the origin of co-ordinates be taken at the starting point, and let distances downward be positive. At any point of the curve, whose ordinate is y , the force of gravity being denoted by g , we have, for the tangential component of gravity, $g \sin \theta$, or, $g \frac{dy}{ds}$; placing this equal to its value, equation (31), we have,

$$g \frac{dy}{ds} = \frac{d^2 s}{dt^2}, \text{ or } g dy = \frac{ds \cdot d^2 s}{dt^2}.$$

Integrating and reducing, remembering that the constant is 0 under the particular hypothesis, we have,

$$gy = \frac{1}{2} \frac{ds^2}{dt^2}; \text{ or, } v^2 = 2gy. \therefore v = \sqrt{2gy} \dots \dots (1)$$

The second member of (1) is the velocity due to the height y . Hence, the velocity generated by a body rolling down a curve, gravity being constant, is equal to that generated by falling freely through the same vertical height.

This principle is true so long as g is constant. But g may be regarded as constant from element to element, no matter what may be the law of variation. Hence, if a body fall toward the sun, or earth, on a spiral line, the ultimate velocity will be the same as though it had fallen on a right line toward the centre of the attracting body. The direction of the motion, however, is not the same, for in the former case it is tangential to the trajectory pursued, and in the latter case it is normal to the attracting body.

The Simple Pendulum.

100. A simple pendulum is a material point suspended from a horizontal axis, by a line without weight, and free to vibrate about that axis.

Let ABC be the arc through which the vibration takes place, and denote its radius by l . The angle CDA is the *amplitude* of vibration; half this angle, ADB , denoted by α , is the *angle of deviation*; and l is the *length of the pendulum*. If the point start from rest, at A , it will, on reaching any point, H , of its path, have a velocity, v , due to the height EK , denoted by y . Hence,

$$v = \sqrt{2gy}.$$

If we denote the angle HDB by θ , we have $DK = l \cos \theta$; we also have $DE = l \cos \alpha$; and since y is equal to $DK - DE$, we have,

$$y = l(\cos \theta - \cos \alpha),$$

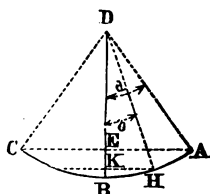


Fig. 27.

which, being substituted in the preceding formula, gives,

$$v = \sqrt{2gl(\cos\theta - \cos\alpha)}.$$

Again, denoting the angular velocity at the time t by φ , and remembering that $\varphi = \frac{d\theta}{dt}$, we have, for the velocity of the pendulum,

$$v = l \times \varphi = l \frac{d\theta}{dt}.$$

Equating the two values of v , and reducing, we have,

$$dt = \sqrt{\frac{l}{2g}} \cdot \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Developing $\cos\theta$ and $\cos\alpha$ by McLaurin's Formula, we have,

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \text{etc.}$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} - \text{etc.}$$

If we suppose the amplitude to be small, we may neglect all the terms after the second; doing so, and substituting in (31), we have,

$$dt = \sqrt{\frac{l}{g}} \cdot \frac{d\theta}{\sqrt{\alpha^2 - \theta^2}}.$$

Integrating between the limits $\theta = -\alpha$, and $\theta = +\alpha$, we find,

$$t = \pi \sqrt{\frac{l}{g}}.$$

which is the formula for the time of vibration of a simple pendulum.

Attraction of Homogeneous Spheres.

101. The Newtonian Law of universal gravitation may be expressed as follows, viz.: every particle of matter attracts every other particle, with a force that varies *directly* as the mass of the attracting particle, and *inversely* as the square of the distance between the particles. To apply this law to the case of homogeneous spheres, let us first consider the action of a spherical shell, of infinitesimal thickness, on a material point within it.

Let D be the material point, APB a great circle through it, and let the diameter ADB be taken as the axis of x . If the circle be revolved about AB , its circumference will generate a spherical shell, and any element of the circumference, as P , will generate an elementary zone whose altitude is dx . For any point of this zone, as P , there is another point, P' ,

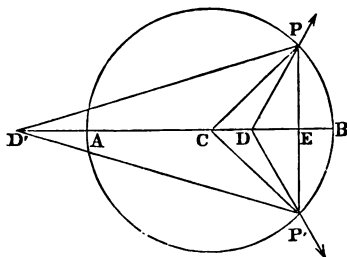


Fig. 28.

symmetrical with it, and the resultant action of these points on D is directed along AB , and is equal to the sum of the forces into the cosine of the angle BDP . Hence, the resultant attraction of the entire zone is directed along AB , and is equal to the sum of the attractions of all its particles into the cosine of the angle BDP . To find an expression for this resultant, denote CP by r , CD by a , DP by z , and CE by x . Because the shell is infinitesimal in thickness and homogeneous, the mass of the zone, is to the mass of the shell, as dx , the altitude of the zone, is to $2r$, the altitude of the shell. Hence, if the mass of the shell be

denoted by m , the mass of the zone will be denoted by $\frac{m dx}{2r}$. If the force exerted by the unit of mass at the unit of distance be taken as the unit of force, the attraction exerted by all the particles of the zone on the point D will be equal to $\frac{m dx}{2r} \cdot \frac{1}{z^2}$, and the resultant action on D , in the direction of AB , denoted by df , will be given by the equation,

$$df = \frac{m dx}{2r} \cdot \frac{1}{z^2} \cdot \cos EDP \dots \dots (1)$$

From the triangle, PCD , we have,

$$\overline{DP}^2 = z^2 = r^2 + a^2 - 2ax = r^2 - a^2 - 2a(x - a),$$

and from the triangle, EDP , we have,

$$\cos EDP = \frac{x - a}{z}.$$

Substituting in (1), remembering that dx equals $d(x - a)$, we have,

$$df = \frac{m}{2r} \cdot \frac{(x - a) d(x - a)}{[r^2 - a^2 - 2a(x - a)]^{\frac{3}{2}}} \dots \dots (2)$$

Regarding $(x - a)$ as a single variable, reducing by Formula A , and integrating by Formula (29), we find,

$$F = \frac{m}{2r} \left\{ - \frac{x - a}{a[r^2 - a^2 - 2a(x - a)]^{\frac{1}{2}}} + \frac{r^2 - a^2}{a^2[r^2 - a^2 - 2a(x - a)]^{\frac{1}{2}}} \right\}.$$

Taking the integral from $x = -r$ to $x = +r$, we have,

$$f'' = 0$$

Hence, the effect of the attraction of the shell on any point within it is null. If a sphere be described about O as a centre with a radius equal to a , we may call that part which lies between it and the surface of the given sphere the *exterior shell*, and the sphere itself may be called the *nucleus*. From what precedes, we infer that any point within a homogeneous sphere is acted on by the sphere precisely as though the exterior shell did not exist. Hence, a point at the centre of a sphere is not affected by the attraction of the sphere.

If the point D' be taken without the shell, we have,

$$\overline{D'P}^2 = z^2 = r^2 + a^2 + 2ax = r^2 - a^2 + 2a(x + a),$$

and from the triangle, $ED'P$, we have,

$$\cos ED'P = \frac{x + a}{z}.$$

Substituting in (1), we have,

$$df = \frac{m}{2r} \frac{(x + a)d(x + a)}{[r^2 - a^2 + 2a(x + a)]^{\frac{3}{2}}} \dots \dots (3)$$

Reducing and integrating as before, we find,

$$f = \frac{m}{2r} \left\{ \frac{x + a}{a[r^2 - a^2 + 2a(x + a)]^{\frac{1}{2}}} + \frac{r^2 - a^2}{a^2[r^2 - a^2 + 2a(x + a)]^{\frac{1}{2}}} \right\} + C.$$

Taking the integral between the limits $x = -r$ and $x = +r$, we have, when $a > r$

$$f'' = \frac{m}{a^2} \dots \dots (4)$$

But m is the mass of the shell, and a is the distance of D from the centre; hence, the shell attracts a particle without it as though its entire mass were concentrated at its centre.

A homogeneous sphere may be regarded as made up of spherical shells; hence, a sphere attracts any point without it, as though the mass of the sphere were concentrated at its centre. The same is true of a sphere made up of homogeneous strata, which vary in density in passing from the surface to the centre. It is to be inferred that two homogeneous spheres, or two spheres made up of homogeneous strata, attract each other as though both were concentrated at their centres.

If an opening were made from the surface to the centre of the earth, supposed homogeneous, and a body were to move along it under the earth's attraction, it would everywhere be urged on by a force varying *directly* as the distance of the body from the centre. For, denote the distance of the body from the centre at any instant by x . The body will only be acted on by the nucleus whose radius is x . If we take r to represent the radius of the earth, and remember that the masses are proportional to these volumes, we have,

$$M : m :: \frac{4}{3}\pi r^3 : \frac{4}{3}\pi x^3 :: r^3 : x^3.$$

Denoting the force of attraction at the surface by g , and the force of attraction at the point whose distance from the centre is x by f , we have, from the Newtonian law,

$$g : f :: \frac{r^3}{r^2} : \frac{x^3}{x^2} :: r : x; \quad \therefore f = \frac{g}{r}x.$$

Hence, the proposition is proved. (See Art. 96.)

Orbital Motion.

102. If a moving point, P , be continually acted on by a deflecting force directed toward a fixed centre, it will describe a line or path, called an *orbit*. If the *moving* point be undisturbed by the action of any other force, the orbit will lie in a plane passing through the fixed centre and an element of the curve. Let this plane be taken as the co-ordinate plane, let the fixed point be the origin, and let the orbit be represented by APB .

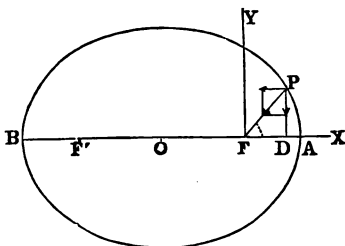


Fig. 29.

Denote the acceleration due to the deflecting force at any time t by f , its inclination to the axis by ϕ , and its components in the directions of the co-ordinate axes by X and Y .

From the figure, we have,

$$X = -f \cos \phi, \text{ and } Y = -f \sin \phi \dots \dots (1)$$

If we regard t as the independent variable, dx and dy will both be variable, and from Equation (3), Art 93, we have,

$$X = \frac{d^2x}{dt^2}, \text{ and } Y = \frac{d^2y}{dt^2}.$$

If we denote the co-ordinates of P by x and y , and its radius vector FP by r , we have, from the figure,

$$\cos \phi = \frac{x}{r}, \sin \phi = \frac{y}{r};$$

and $x^2 + y^2 = r^2; \therefore xdx + ydy = rdr.$

Substituting in (1), we have,

$$\frac{d^2x}{dt^2} = -f \frac{x}{r}, \text{ and } \frac{d^2y}{dt^2} = -f \frac{y}{r} \dots\dots (2)$$

Multiplying the first of Equations (2) by y , the second by x , and subtracting the former from the latter, we have,

$$\frac{xd^2y}{dt^2} - \frac{yd^2x}{dt^2} = 0, \text{ or } \frac{d(xdy - ydx)}{dt} = 0 \dots\dots (3)$$

Multiplying the first by dx , the second by dy , adding the resulting equations and reducing, we have,

$$\frac{dx d^2x}{dt^2} + \frac{dy d^2y}{dt^2} = -f \frac{xdx + ydy}{r} = -f dr \dots\dots (4)$$

Integrating (3) and (4), Art. 77, we have,

$$x \frac{dy}{dt} - y \frac{dx}{dt} = C \dots\dots (5)$$

and

$$\frac{1}{2} \frac{dx^2 + dy^2}{dt^2} + C' = -\int f dr;$$

or,

$$\frac{ds^2}{dt^2} + 2\int f dr = C'' \dots\dots (6)$$

Equations (5) and (6) make known the circumstances of motion when the value of f is given. It is found convenient to transform them to a system of polar co-ordinates, whose pole is F , and whose initial line is FX . The formulas for transformation are

$$x = r \cos \phi, \text{ and } y = r \sin \phi \dots\dots (7)$$

Hence, by differentiating, we have,

$$\frac{dy}{dt} = \frac{dr}{dt} \sin \phi + r \cos \phi \frac{d\phi}{dt} \dots\dots (8)$$

and

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \phi - r \sin \phi \frac{d\phi}{dt} \dots\dots (9)$$

We also, have, Art. (53),

$$ds^2 = r^2 d\varphi^2 + dr^2 \dots \dots (10)$$

Substituting in (5), and reducing by the relations,

$$x \sin \varphi - y \cos \varphi = 0, \text{ and } x \cos \varphi + y \sin \varphi = \frac{x^2 + y^2}{r} = r,$$

we have,

$$\frac{r^2 d\varphi}{dt} = C \dots \dots (11)$$

From equation (6), we have,

$$\frac{dr^2 + r^2 d\varphi^2}{dt^2} + 2 \int f dr = C'' \dots \dots (12)$$

But, from (11), we have,

$$dt = \frac{r^2 d\varphi}{C}, \text{ or, } dt^2 = \frac{r^4 d\varphi^2}{C^2},$$

and this, in (12), gives,

$$C^2 \left\{ \frac{dr^2}{r^4 d\varphi^2} + \frac{1}{r^2} \right\} + 2 \int f dr = C'' \dots \dots (13)$$

Equations (11) and (13) are the equations required.

Multiplying (11) by dt , and integrating, we have,

$$\int r^2 d\varphi = Ct + C''' \dots \dots (14)$$

But, by Art. 53, $r^2 d\varphi$ is twice the elementary area swept over by the radius vector; hence, the first member of (14) is an expression for twice the area swept over by the radius vector up to the time t . If we suppose the area to be reckoned from the initial line FX , and at the same time

suppose $t = 0$, we have $C''' = 0$. Substituting this in (14), and in the resulting equation making $t = 1$, we find C equal to twice the area swept over in the first unit of time; denoting the area described in the unit of time by A , we have, $C = 2A$, and consequently,

$$\int r^2 d\phi = 2At \dots\dots (15)$$

From Equation (15) we infer that the areas described by the radius vector are proportional to the times of description, and this without reference to the nature of the deflecting force.

To find the equation of the orbit, let us assume as a particular case, the Newtonian law of universal gravitation. Denoting the force exerted by the central body at a unit's distance by k , we have, for the attraction at the distance r ,

$$f = \frac{k}{r^2} \dots\dots (16)$$

Making $r = \frac{1}{u}$, whence $dr = -\frac{du}{u^2}$, and substituting for C and f their values in (13), we have,

$$4A^2 \left\{ \frac{du^2}{d\phi^2} + u^2 \right\} - 2ku = C^{iv} \dots\dots (17)$$

or,

$$\frac{du^2}{d\phi^2} + u^2 = \frac{2k}{4A^2}u + C^{iv} \dots\dots (18)$$

To find the value of C^{iv} , let $\frac{du}{d\phi} = 0$, when $r = r'$, or $u = u'$. But when $\frac{du}{d\phi} = 0$, we also have $\frac{dr}{d\phi} = 0$, that is,

the radius vector is perpendicular to the arc. If we denote the corresponding velocity by v' , we have,

$$A = \frac{v'r'}{2}, \text{ or, } 4A^2 = v'^2 r'^2;$$

and, consequently,

$$\frac{2k}{4A^2}u = \frac{2k}{v'^2 r'^2}.$$

Making these substitutions in (18), we find,

$$C^v = \frac{1}{r'^2} - \frac{2k}{v'^2 r'^3}.$$

Denoting this value of C^v by S , and solving Equation (18), we have,

$$\begin{aligned} \frac{du^2}{d\varphi^2} = S - \left(u^2 - \frac{2k}{4A^2}u\right) &= S + \frac{k^2}{16A^4} - \left(u - \frac{k}{4A^2}\right)^2 \\ &= p^2 - (u - q)^2. \end{aligned}$$

Solving with reference to φ , and taking the negative sign of the radical, we have,

$$d\varphi = - \frac{du}{\sqrt{p^2 - (u - q)^2}} = - \frac{d(u - q)}{\sqrt{p^2 - (u - q)^2}} \dots (19)$$

Integrating, we have,

$$\varphi - \alpha = \cos^{-1} \frac{u - q}{p} \dots (20)$$

In which $-\alpha$ is an arbitrary constant. If we suppose $\varphi = 0$, when $v = v'$, $r = r'$, and $u = u'$, the corresponding value of α is the angular distance from the initial line to the radius vector whose direction is perpendicular to the element of the curve.

Taking the cosines of both members of (20), we have,

$$\frac{u - q}{p} = \cos(\varphi - \alpha), \text{ or, } u = q + p \cos(\varphi - \alpha)$$

and finally replacing u by its value, and solving, we have,

$$r = \frac{1}{q + p \cos(\varphi - \alpha)} = \frac{4A^2}{k + \sqrt{16SA^2 + k^2} \times \cos(\varphi - \alpha)} \dots\dots (21)$$

Equation (21) is the polar equation of a conic section. Hence, the orbit of a particle about a central attracting body is one of the conic sections. If the orbit is a closed curve, it must be an ellipse. This corresponds to the case of a planet revolving about the sun, or to that of a satellite revolving about its primary.

Law of Force.

103. If the orbit of a revolving particle be an ellipse, it must be attracted to the focal point by a force that varies inversely as the square of the distance.

The polar equation of an ellipse is

$$r = \frac{a(1 - e^2)}{1 + e \cos(\varphi - \alpha)} \dots\dots (1)$$

in which a is the semi-transverse axis, e is the eccentricity, α is the angular distance from the fixed line to the radius vector drawn to the nearest vertex, or the longitude of the perihelion in the case of a planetary orbit. The angle, $\varphi - \alpha$, is the angle that is called in astronomy the true anomaly.

Putting $r = \frac{1}{u}$, in (1), we get,

$$u = \frac{1 + e \cos(\varphi - \alpha)}{a(1 - e^2)} \dots\dots (2)$$

Differentiating twice, with respect to φ , we find,

$$\frac{du}{d\varphi} = -\frac{e \sin(\varphi - \alpha)}{a(1 - e^2)} \dots \dots (3)$$

and,

$$\frac{d^2u}{d\varphi^2} = -\frac{e \cos(\varphi - \alpha)}{a(1 - e^2)} \dots \dots (4)$$

Adding (2) and (4), we have,

$$\frac{d^2u}{d\varphi^2} + u = \frac{1}{a(1 - e^2)} \dots \dots (5)$$

Resuming equation (13) of the last article, replacing r by its value, $\frac{1}{u}$, and dr by its value, $-\frac{du}{u^2}$, differentiating and reducing, we have,

$$C^2 \left\{ \frac{d^2u}{d\varphi^2} + u \right\} - \frac{f}{u^2} = 0 \dots \dots (6)$$

Combining (5) and (6), and solving with respect to f , we have,

$$f = \frac{C^2 u^2}{a(1 - e^2)}, \text{ or } f = \frac{C^2}{a(1 - e^2)} \cdot \frac{1}{r^2} \dots \dots (7)$$

Hence, f varies inversely as the square of r , which was to be shown.

Note on the Methods of the Calculus.

104. All the rules and principles of the Calculus, in the present treatise, have been deduced in accordance with *the method of infinitesimals*, as explained in Articles 6 and 7. They might also have been deduced by *the method of limits*. It remains to be shown that the results obtained by these two methods are always *identically* the same.

To explain the *method of limits*, let us denote any function of x by y ; that is, let us assume the equation,

$$y = f(x) \dots\dots (1)$$

If we increase x by a variable increment h , and denote the corresponding value of y by y' , we have,

$$y' = f(x + h) \dots\dots (2)$$

It is shown in Courtenay's Calculus, Article 4, that so long as x retains its general value, the new state of the function can be expressed by the formula,

$$y' = f(x + h) = f(x) + Ah + Bh^2 + Ch^3 + \text{etc.} \dots (3)$$

in which A, B, C , etc., depend on x , but are independent of h . If we subtract (1) from (3), member from member, we have,

$$y' - y = Ah + Bh^2 + (\text{etc.}) h^3 \dots\dots (4)$$

Dividing both members of (4) by h , we have,

$$\frac{y' - y}{h} = A + Bh + (\text{etc.}) h^2 \dots\dots (5)$$

The first member of (5) is a symbol to express the ratio of the increment of the variable, to the corresponding increment of the function, and the second member is the value of that ratio.

It is shown in Algebra, that in an expression like the second member of equation (5), it is always possible to give to h a value small enough to make the first term numerically greater than the algebraic sum of all the others. If we assign such a value to h , and then suppose h to go on diminishing, the second member will continually approach A , and when h becomes 0, the second mem-

ber will reduce to A . Hence, A is a quantity toward which the ratio, $\frac{y' - y}{h}$, approaches as h is diminished, but beyond which it cannot pass; it is therefore the *limit* of that ratio. This limit is *the differential coefficient of y* , and, as may easily be seen, it is entirely independent of dx . The product of this by the differential of the variable is *the differential of y* .

We may, therefore, enunciate *the method of limits* as follows: viz., *Give to the independent variable a variable increment, and find the corresponding state of the function; from this subtract the primitive state, and divide the difference by the variable increment; then pass to the limit of the quotient by making the increment of the variable equal to 0; the result is the differential coefficient of the function; if this be multiplied by the differential of the variable, the product is the differential of the function.*

In the case assumed, we have, by this method,

$$\frac{dy}{dx} = A; \therefore dy = A dx \dots \dots (6)$$

If we make $h = dx$, in Equation (4), dx being infinitely small, the first member will be the differential of y , and all the terms of the second member after the first may be neglected. Hence, by the method of infinitesimals, we have,

$$dy = A dx; \therefore \frac{dy}{dx} = A.$$

This result is the same as that obtained by the method of limits. But, by hypothesis, y represents any function of x ; hence, in all cases, the differential coefficient is identically the same whether found by the method of limits or by the method of infinitesimals.

Every function, regarded as a *primitive*, is connected with some other function, regarded as a *derivative*, by the law of differentiation. This derivative is the differential coefficient of the primitive. The object of the differential calculus is to find the derivative from its primitive; the object of the integral calculus is to find the primitive from its derivative; every application of the calculus depends on one of these processes, or on some discussion growing out of one, or the other. Now, because the primitive and the derivative are independent of the differentials of both function and variable, the relation between them is, of necessity, independent of the methods employed in establishing that relation. But it has been shown that the same relation is found between these functions, whether we employ the method of infinitesimals or the method of limits. Hence, these methods, and the results obtained by them, are in all cases logically identical.

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During the period alluded to, many authors and editors in this department have started into public notice, and by borrowing ideas and processes original with Dr. Davies, have enjoyed a brief popularity, but are now almost unknown. Many of the series of to-day, built upon a similar basis, and described as "modern books," are destined to a similar fate; while the most far-seeing eye will find it difficult to fix the time, on the basis of any data afforded by their past history, when these books will cease to increase and prosper, and fix a still firmer hold on the affection of every educated American.

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Davies' National Course of Mathematics.

TESTIMONIALS.

From L. VAN BOKKELEN, State Superintendent Public Instruction, Maryland.

The series of Arithmetics edited by Prof. Davies, and published by your firm, have been used for many years in the schools of several counties, and the city of Baltimore, and have been approved by teachers and commissioners.

Under the law of 1865, establishing a uniform system of Free Public Schools, these Arithmetics were unanimously adopted by the State Board of Education, after a careful examination, and are now used in all the Public Schools of Maryland.

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From HORACE WEBSTER, President of the College of New York.

The undersigned has examined, with care and thought, several volumes of Davies' Mathematics, and is of the opinion that, as a whole, it is the most complete and best course for Academic and Collegiate instruction, with which he is acquainted.

From DAVID N. CAMP, State Superintendent of Common Schools, Connecticut.

I have examined Davies' Series of Arithmetics with some care. The language is clear and precise; each principle is thoroughly analyzed, and the whole so arranged as to facilitate the work of instruction. Having observed the satisfaction and success with which the different books have been used by eminent teachers, it gives me pleasure to commend them to others.

From J. O. WILSON, Chairman Committee on Text-Books, Washington, D. C.

I consider Davies' Arithmetics decidedly superior to any other series, and in this opinion I am sustained, I believe, by the entire Board of Education and Corps of Teachers in this city, where they have been used for several years past.

From JOHN L. CAMBELL, Professor of Mathematics, Wabash College, Indiana.

A proper combination of abstract reasoning and practical illustration is the chief excellence in Prof. Davies' Mathematical works. I prefer his Arithmetics, Algebras, Geometry and Trigonometry to all others now in use, and cordially recommend them to all who desire the advancement of sound learning.

From MAJOR J. H. WHITTLESEY, Government Inspector of Military Schools.

Be assured, I regard the works of Prof. Davies, with which I am acquainted, as by far the best text-books in print on the subjects which they treat. I shall certainly encourage their adoption wherever a word from me may be of any avail.

From T. McC. BALLANTINE, Prof. Mathematics Cumberland College, Kentucky.

I have long taught Prof. Davies' Course of Mathematics, and I continue to like their working.

From JOHN MCLEAN BELL, B. A., Prin. of Lower Canada College.

I have used Davies' Arithmetical and Mathematical Series as text-books in the schools under my charge for the last six years. These I have found of great efficacy in exciting, invigorating, and concentrating the intellectual faculties of the young.

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From D. W. STEELE, Prin. Philoioian Academy, Cold Springs, Texas.

I have used Davies' Arithmetics till I know them nearly by heart. A better series of school-books never were published. I have recommended them until they are now used in all this region of country.

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It will be observed that the above are all particulars in which School Histories have been signally defective, or altogether wanting. Many other claims to favor it shares in common with its predecessors.

The National Series of Standard School-Books.

BARNES' ONE-TERM HISTORY—Continued.

From PROF. WM. F. ALLEN, State Univ. of Wisconsin.

I think the author of the new "Brief History of the United States" has been very successful in combining brevity with sufficient fullness and interest. *Particularly*, he has avoided the excessive number of names and dates that most histories contain. Two features that I like *very much* are the *anecdotes* at the foot of the page and the "*Historical Recreations*" in the Appendix. The latter, I think, is quite a *new* feature, and the other is *very* well executed.

From S. G. WRIGHT, Assist.-Supt. Pub. Inst., Kansas.

It is with extreme pleasure we submit our recommendation of the "Brief History of the United States." It meets the needs of young and older children, combining concision with perspicuity, and if "brevity is the soul of wit," this "Brief History" contains not only that well-chosen ingredient, but wisdom sufficient to enlighten those students who are wearily longing for a "new departure" from certain old and uninteresting presentations of fossilized writers. We congratulate a progressive public upon a progressive book.

From HON. NEWTON BATEMAN, Supt. Pub. Inst., Illinois.

Barnes' One-Term History of the United States is an exceedingly attractive and spirited little book. Its claim to several new and valuable features seems well founded. Under the form of six well-defined Epochs, the History of the United States is traced tersely, yet pithily, from the earliest times to the present day. A good map precedes each epoch, whereby the history and geography of the period may be studied together, *as they always should be*. The syllabus of each paragraph is made to stand in such bold relief, by the use of large, heavy type, as to be of much *mnemonic* value to the student. The book is written in a sprightly and piquant style, the interest never flagging from beginning to end—a rare and difficult achievement in works of this kind.

From the "Chicago Schoolmaster" (Editorial).

A thorough examination of Barnes' Brief History of the United States brings the examiner to the conclusion that it is a superior book in almost every respect. The book is neat in form, and of good material. The type is clear, large, and distinct. The facts and dates are correct. The arrangement of topics is just the thing needed in a history text-book. By this arrangement the pupil can see at once what he is expected to do. The topics are well selected, embracing the leading ideas or principal events of American history. The book as a whole is much superior to any I have examined. So much do I think this, that I have ordered it for my class, and shall use it in my school.

(Signed) B. W. BAKER.

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This book is written in the style used by a father talking with his children on the progress of history. As one Age after another is taken up, the author brings before the young reader the prominent men and characteristic events by which it is to be remembered. The object is to stimulate the pupil in school or the child at home to study history, to think of it as a lively picture of the doings of men, and not as a dead list of uninteresting dates.

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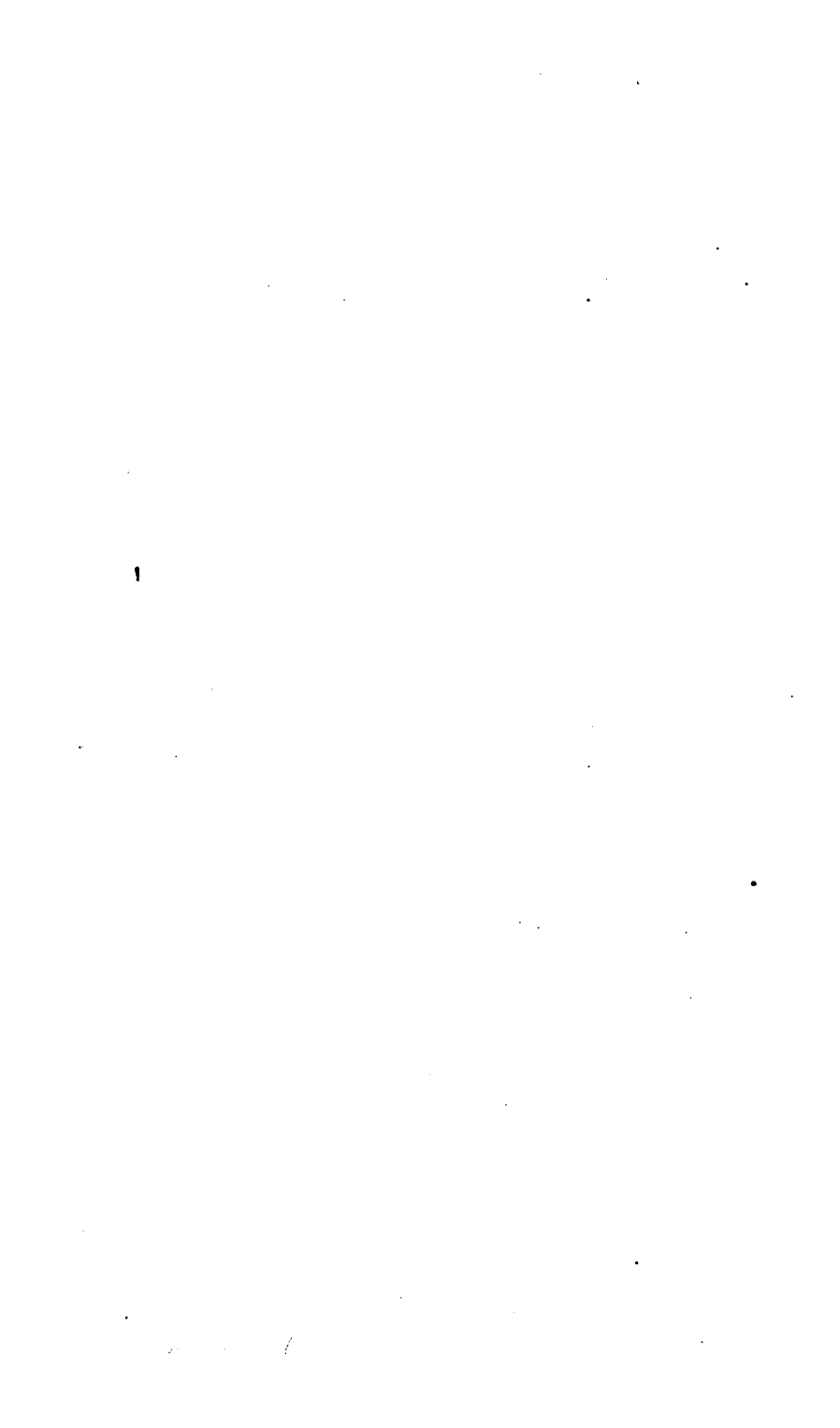
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